Abstract

The development of closed loop control has been one of the major technological achievements of the twentieth century, and its concepts married with analogue and digital computation led to explosive growth in the sophistication of the technology and the ubiquity of its application. By that, an *automatic flight control system* is able to stabilize an inherently unstable airplane or to provide a *stability augmentation system (SAS)*. In other words, there is no margin for error in its response to atmospheric disturbances and to commands that come from navigation system. Moreover it improves the handling qualities, the flight comfort and reduces the costs when an airplane is under control of an autopilot.

This specific project is destined to simulate and analyze the operating parameters (i.e., altitude, pitch angle, airspeed, throttle level, and elevator angle) of a passenger aircraft -Boeing *747-100* –in longitudinal motion deflected from its ordered flight trajectory and atmospheric condition when related with its reference steady flight parameters.

For all cases studied, the results are satisfactory controlled and expected. Future works can be done in using the differential equations instead of linearized ones, which increases the range of possible cases to be studied.

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1. Glossary

α	angle of attack
A	resultant aerodynamic force vector acting on the airplane, in F_B
c	control vector
C_L	$Lift\ Force \big/ \frac{1}{2}\rho V^2 S$
C_D	$Drag\ Force \big/ \frac{1}{2}\rho V^2 S$
CG	center mass of the airplane
$\bar{c}$	length of mean aerodynamic chord
f	resultant external force vector
F_B	body frame
F_E	earth frame
g	acceleration due to gravity
G	resultant external moment vector, about the mass center
h	altitude
h	angular momentum vector of the airplane with respect to its mass center
(h_x, h_y, h_z)	scalar components of **h** in F_B
I_x, I_y, I_z	moments of inertia about (x, y, z) axes
I_{yz}, I_{xz}, I_{xy}	products of inertia $\int yz\ dm, \int xz\ dm, \int xy\ dm$, respectively
(L, M, N)	scalar components of **G** in F_B
m	airplane mass

M	mach number
(p, q, r)	scalar components of $\boldsymbol{\omega}$, radians/sec in F_B
S	wing area
(u, v, w)	scalar components of **V** in F_B
Γ	vertical wind gradient, $-dW/dh$
T	thrust force
V	airspeed vector of airplane mass center
x_E, y_E, z_E	coordinates of airplane mass center relative to fixed axes
(X, Y, Z)	components of resultant aerodynamic force acting on the airplane, in F_B
δ_e	elevator angle
δ_p	propulsion control
$\boldsymbol{\omega}$	angular velocity vector of the airplane
W	wind speed vector, in F_E
(ψ, θ, ϕ)	Euler angles, radians

see also Tab. 4.1, Tab. 4.2 and Tab. 4.3.

see also Sec. 5.5.

2. Preface

2.1. Origin of the project

Our great heroes of aviation, the inventors of the aircraft, are all the pioneers since da Vinci to the Priest Bartholomeu de Gusmão, Richard Pearse, Clement Ader, the Americans brothers Wright, Alberto Santos-Dumont, Sikorky and Fokker.

It was in that time, almost 100 years ago, where a man flied, composing the main goals of the flight: took-off, remained itself in level and landed, using its proper propulsive ways. For example, was in October, 23, 1906 that our “petit Santos-Dumont” got these challenges through a fabric airplane – 14-BIS.

Due to challenges faced in the aeronautical field, during from the earlier of the last century until nowadays, many research, simulation and works were carried out to provide the quality, security and the viability of the flight trip and aircrafts.

2.2. Motivation

A flight simulator is a system in which a real pilot is with the orders of a virtual plane whose behavior is obtained by simulation. They are largely used by the airline companies, the aircraft industry military and civil for the formation continues pilots (new type of planes or equipment, extreme situations, military operations) and to develop new planes.

Inside one flight simulator and the developing of new planes, this project participates on the understanding and on the exploration of the behavior of the Boeing 747-100 in adverse flight condition from the actions of an automatic control system.

2.3. Previous Requirement

It is desirable that the reader have some basic knowledge in mathematics (i.e., Laplace transform and matrix operation), in automatic control theory (block-diagram representation and transfer function), and in fluid mechanics concepts.

In addition, for the simulation, it is pleasing some familiarity with Simulink / Matlab.

3. Introduction

Since the end of 20th century, the search for mechanical systems and devices with high autonomy increased significantly. Also, cost reduction, security, smooth and an efficient service, between many others, were and are the differentials demanded for these products and services in a complex and competitive business environment.

In addition, an important attention inside the engineering world should be done in those systems where thousands of hundreds people inside of thousands euros are involved, and crossing skies and barriers. They are the *Passenger Aircrafts*.

According to [2], the sustained air traffic growth continued (4.1% on average) in 2006 with an important remark for “Low-Cost” and Business Aviation, where their traffic shares were 16% and 7%. Totally in Europe during 2006, there were 9.6 million of flights per year.

Important to remark that, even with these actual statistics, it reinforces the importance of those aspects mentioned at the beginning.

As a result of this, the development of an automatic control system was inevitable. The closed loop control, and later, its concepts joined with the analogue and digital computation let to an explosive growth in the sophistication of the technology and the ubiquity of its application.

For example, an *automatic flight control system* is able to adjust the operating parameters of an aircraft in order to follow the commands that come from the navigation system, even in the presence of atmospheric disturbances.

Moreover, it is capable to stabilize an inherently unstable airplane (even during a landing process, where it becomes more sensible to external interferences), to provide a *stability augmentation system (SAS)*, to improve the handling qualities and flight comfort, amongst others.

3.1. Objective of the Project

Based on this, the present report is destined to simulate and analyze the behavior of necessaries operating parameters (i.e., flight altitude, pitch angle, airspeed, throttle level, and elevator angle) of a passenger aircraft –Boeing 747-100- to guarantee its dynamic longitudinal stability on adverse atmosphere and situation conditions.

In more details, trying to cover important and usual cases, where the auto-pilot is desirable, these situation conditions, mentioned above, refer to *Cruising Flight Phase* and *Approximation Flight Phase* for landing.

Furthermore, two different cases will be considered for the atmosphere on the last flight situation (landing approach). They are: exponential wind speed profile and sinusoidal wind speed profile, which symbolize the interaction due to roughness existent between the earth and the airflow.

The results are obtained by simulation carried out in *Simulink/Matlab*, and the models, on it developed, are based on the block-diagram representation. Specifically, the aircraft model is achieved throughout the linearized longitudinal motion equations.

3.2. Scope of the Project

The application of this specific project is destined for those who work on the research of aeronautics or automation field. These both groups are beneficed through many different possible simulation cases, by varying some parameters (i.e., initial flight conditions, control and airplane characteristics, among others).

Also, it may be used as an interactive software destined for those passionate on the aeronautics field without deep familiarity on it, in which they can, by simulating different initial flight conditions and assimilating the respective operating parameters, develop their knowledge and interest on this fascinating Flight Vehicle Environment.

4. General Equations of Unsteady Motion

4.1. General Remarks

The basis for analysis, computation, or simulation of the unsteady motions of a flight vehicle is the mathematical model of the vehicle and its subsystems. Obviously, an airplane in flight is a very complicated dynamic system. It consists of an aggregate of elastic bodies so connected that both rigid and elastic relative motions can occur. Also, the external forces that act on the airplane are also complicated functions of its shape and its motion. It seems clear that realistic analyses of engineering precision are not likely to be accomplished with a very simple mathematical model.

The model developed in the following of this chapter, has been found by aeronautical engineers and researchers to be very useful in practice.

The vehicle is treated as a single rigid body with six degrees of freedom. This body is free to move in the atmosphere under the actions of gravity and aerodynamic forces- it is primarily the nature and complexity of aerodynamic forces that distinguish flight vehicles from other dynamic system.

4.2. The Rigid-Body Equations

In the interest of completeness, the rigid-body equations are derived from first principles of Newton. Into an element of the airplane *dm* is applied the Newton's laws, and then integrate over all elements. The velocities and accelerations must of course be relative to an inertial, or Newtonian, frame of reference. By considering, the frame F_E, fixed to the Earth, is assumed to be such a frame. Also, the velocities relative to F_E are identified by a superscript E.

By instant, we shall temporarily assume that **W** = 0 (air velocity), and to avoid carrying the superscript E throughout the following development, it is considered that $\mathbf{V}^{\mathbf{E}} = \mathbf{V}$. In the frame F_B (body frame):

$$V_B = [u \quad v \quad w]^T \qquad \text{(Eq. 4.1)}$$

Through the Fig. 4.1, the position vector of *dm*, relative to the origin of F_E, is $\mathbf{r}_c + \mathbf{r}$. In the frame F_E,

$$r_{CE} = [x_E \quad y_E \quad z_E]^T \qquad \text{(Eq. 4.2)}$$

and, in the frame F_B,

$$r_B = [x \quad y \quad z]^T \qquad \text{(Eq. 4.3)}$$

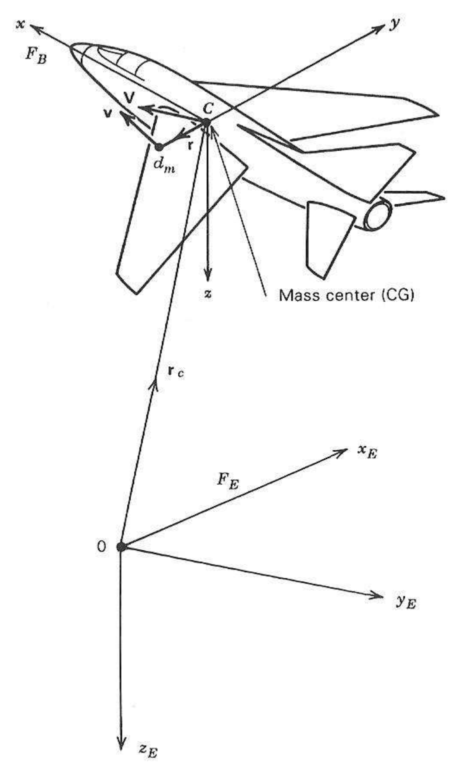

Fig. 4.1. Vectors Position *[1]*

The inertial velocity $\mathbf{v}_E$ and the momentum $\mathbf{p}_E$ of *dm* are

$$\mathrm{v}_E = \left(\dot{\mathbf{r}}_{CE} + \dot{\mathbf{r}}_E \right) = V_E + \dot{\mathbf{r}}_E \quad \text{(Eq. 4.4)}$$

$$\mathrm{p}_E = \mathrm{v}_E \, dm \quad \text{(Eq. 4.5)}$$

The momentum of the whole airplane is

$$\int \mathbf{v}_E dm = \int (\mathbf{V}_E + \dot{\mathbf{r}}_E) dm = \mathbf{V}_E \int dm + \int \dot{\mathbf{r}}_E \, dm \quad \text{(Eq. 4.6)}$$

Since C is the center of mass, the last integral of the Eq. 4.6 is zero. Thus,

$$\int \mathbf{v}_E dm = m\mathbf{V}_E \quad \text{(Eq. 4.7)}$$

where,

m = total mass of the airplane;

Applying the Newton’s second law into a portion of mass *dm*, we obtained the resultant of all forces acting on it. $\mathbf{f}_E$ is the summation of these forces upon all the elements on the airplane. So,

$$\mathbf{f}_E = \int d\mathbf{f}_E = \int \dot{\mathbf{v}}_E \, dm \quad \text{(Eq. 4.8)}$$

From Eq. 4.7, the equation Eq. 4.8 becomes

$$\mathbf{f}_E = m \dot{\mathbf{V}}_E \quad \text{(Eq. 4.9)}$$

This equation relates the external force on the airplane to its movement of the CG, once that, by the Newton’s third law, the internal forces occur in equal and opposite pairs, and hence they contribute nothing to the summation of the forces.

To describe completely the motion of the airplane, it is necessary to know the relation between the external moment and its rotation. It can be obtained considering the moment of momentum [6].

The moment of momentum of *dm*, with respect to C, is

$$d\mathbf{h}_E = \mathbf{r}_E \times \mathbf{v}_E dm = \tilde{\mathbf{r}}_E \mathbf{v}_E dm \qquad \text{(Eq. 4.10)}$$

where,

$$\tilde{\mathbf{r}}_E = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$$

Considering that,

$$\frac{d}{dt}\left(d\mathbf{h}_E\right) = \dot{\tilde{\mathbf{r}}}_E\, \mathbf{v}_E dm + \tilde{\mathbf{r}}_E\, \dot{\mathbf{v}}_E\, dm \qquad \text{(Eq. 4.11)}$$

From Eq. 4.4,

$$\dot{\tilde{\mathbf{r}}}_E = \tilde{\mathbf{v}}_E - \tilde{\mathbf{V}}_E$$

and the *moment* of *d***f** about C is

$$d\mathbf{G} = \mathbf{r} \times d\mathbf{f}$$

Considering Eq. 4.8

$$d\mathbf{G}_E = \tilde{\mathbf{r}}_E d\mathbf{f}_E = \tilde{\mathbf{r}}_E\, \dot{\mathbf{v}}_E\, dm \qquad \text{(Eq. 4.12)}$$

and, also from Eq. 4.11

$$d\mathbf{G}_E = \frac{d}{dt}\left(d\mathbf{h}_E\right) - (\tilde{\mathbf{v}}_E - \tilde{\mathbf{V}}_E)\,\mathbf{v}_E\, dm \qquad \text{(Eq. 4.13)}$$

Known that $\mathbf{v}\times\mathbf{v}=0$ and integrating Eq. 4.13 through all the elements of the airplane, it becomes

$$\int d\mathbf{G}_E = \frac{d}{dt}\int\left(d\mathbf{h}_E\right) + \tilde{\mathbf{V}}_E \int \mathbf{v}_E\, dm \qquad \text{(Eq. 4.14)}$$

where,

$\int d\mathbf{G}$ is the resultant external moment about C (similar reason described to the force equation);

$\int d\mathbf{h}$ is called the *moment of momentum,* or *angular momentum* of the airplane;

From Eq. 4.7 and using that $\mathbf{V}\times\mathbf{V}=0$, the Eq. 4.14 reduces to

$$\mathbf{G}_E = \dot{\mathbf{h}}_E \qquad \text{(Eq. 4.15)}$$

where,

$\mathbf{h}_E = \int \tilde{\mathbf{r}}_E\, \mathbf{v}_E\, dm$ - more details for **h** are found in he following of this chapter;

Important conclusions from equations until now are listed below:

- Both **G** and **h** in Eq. 4.15 are referred to a moving point, the mass center;
- For any other moving point, except the mass center, the equation does *not* in general apply;
- Both equations Eq. 4.9 and Eq. 4.15 are valid when there is relative motion of parts of the airplane;

Removing the hypothesis applied in the principle of this section (it implies, wind velocity **W**≠0), then the velocity $\mathbf{V}_E$ in Eq. 4.6 corresponds to that one of the CG relative to F_E, and the angular momentum **h** does not varies, whether **W** is present or not.

Hence, considering a general case when the wind is present, the motion of the airplane is given by six scalar equations simplified in these vector equations written below:

$$\begin{aligned} \mathbf{f}_E &= m\dot{\mathbf{V}}_E^E \\ \mathbf{G}_E &= \dot{\mathbf{h}}_E \end{aligned} \qquad \text{(Eq. 4.16)}$$

where,

$\dot{\mathbf{V}}_E^E$ is the airplane velocity related to frame F_E and expressed on it;

4.3. Evaluation of the Angular Momentum h

Considering Eq. 4.12 and expressing the components of angular momentum in F_B, we have that

$$\mathbf{h}_B = \int \tilde{\mathbf{r}}_B \, \mathbf{v}_B \, dm \qquad \text{(Eq. 4.17)}$$

$\mathbf{v}_B$ is the speed of one point in a rigid rotating body and it is given by

$$\mathbf{v}_B = \mathbf{V}_B + \tilde{\omega}_B \, \mathbf{r}_B \qquad \text{(Eq. 4.18)}$$

where,

ω_B correspond to the angular velocity (p, q and r) of the airplane relative to the inertial space and it rotates through the x, y and z axe, respectively.

$$\omega_B = \begin{bmatrix} p & q & r \end{bmatrix}^T \rightarrow \tilde{\omega}_B = \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix} \quad \text{(Eq. 4.19)}$$

Now, substituting all the equations presented above in this section on Eq. 4.17

$$\mathbf{h}_B = \int \tilde{\mathbf{r}}_B (\mathbf{V}_B + \tilde{\omega}_B \mathbf{r}_B) dm$$

$$= \left(\int \tilde{\mathbf{r}}_B \, dm \right) \mathbf{V}_B + \int \tilde{\mathbf{r}}_B \, \tilde{\omega}_B \, \mathbf{r}_B dm \quad \text{(Eq. 4.20)}$$

Since the origin of **r** is the CG, the first integral of Eq. 4.20 vanishes. Therefore, expanding the other part of this equation, we get the result for $\mathbf{h}_B$:

$$\mathbf{h}_B = \mathbf{I}_B \boldsymbol{\omega}_B \quad \text{(Eq. 4.21)}$$

$\mathbf{I}_B$ is the *inertia matrix* and its elements are the moments and products of inertia of the airplane, showed in the following Eq. 4.22.

$$\mathbf{I}_B = \begin{bmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{yx} & I_y & -I_{yz} \\ -I_{zx} & -I_{zy} & I_z \end{bmatrix} \quad \text{(Eq. 4.22)}$$

Assumptions: the *xz* plane is a plane of symmetry, then $I_{xy}= I_{yz}=0$.

4.4. Orientation and Position of the Airplane

The position and orientation of the airplane is given relative to earth-fixed frame F_E. At first instant (initial condition), the airplane is imagined to be orientated in which manner that its axes are parallel to those of F_E.

Any changes on its orientation (see Fig. 4.2) are given by a series of three consecutives rotations (**Ψ**, **Θ** and **Φ**) described below:

1 – A rotation **Ψ** is given through the oz_1 axe; $-\pi \le \psi < \pi$

2 – A rotation **Θ** is given through the oy_2 axe; $-\pi/2 \le \theta \le \pi/2$

3 – A rotation **Φ** is given through the ox_3 axe; $-\pi \le \phi < \pi$

Remark: These are the Euler angles and its rotation is given around the new axes. For example: in the step 2 the rotation Θ is around the new axe oy_2 displayed after the rotation Ψ (step 1), and with similar manner, the ox_3 axe is created after the rotation Θ.

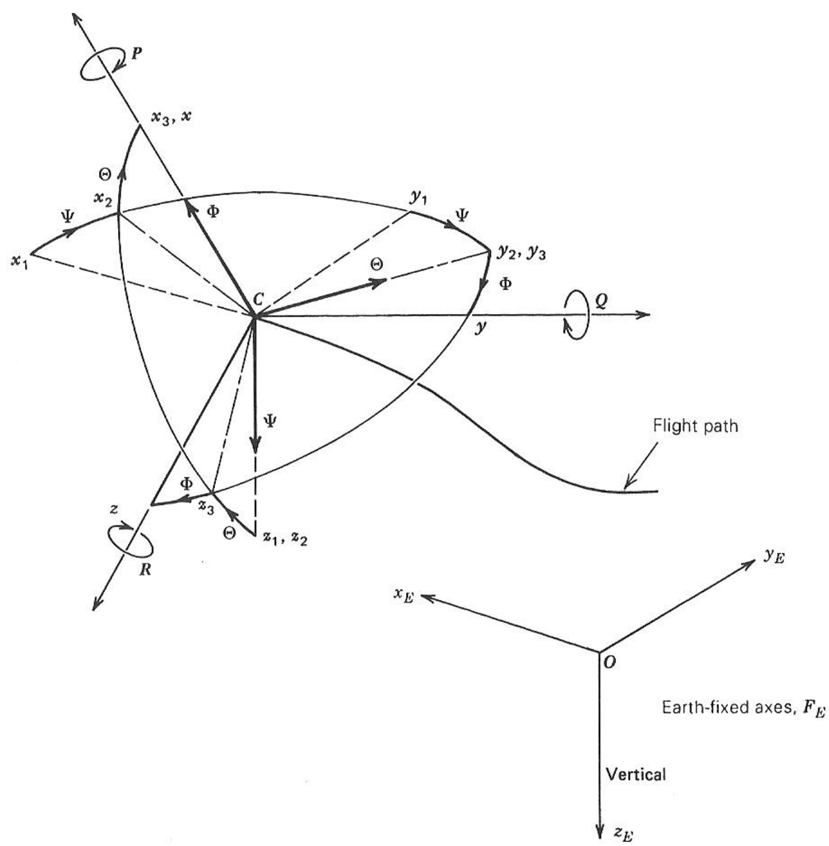

Fig. 4.2. Airplane Orientation *[1]*

4.4.1. The Flight Path

For some parameters in the equations, it is needed to kwon the components of the velocity expressed in both frames. These are getting by transforming the velocity vector $\mathbf{V}_B^E$ into $\mathbf{V}_E^E$ or $\mathbf{V}_E^E$ into $\mathbf{V}_B^E$, using $\mathbf{L}_{EB}$ or $\mathbf{L}_{BE}$ (matrix of direction cosines), respectively. For more details about this transformation, see Appendix A.4. in *[1].*

$$\mathbf{V}_E^E = \mathbf{L}_{EB}\mathbf{V}_B^E \tag{Eq. 4.23}$$

$$\begin{bmatrix} \dot{x}_E \\ \dot{y}_E \\ \dot{z}_E \end{bmatrix} = \mathbf{L}_{EB}\mathbf{V}_B^E$$

where,

$$\mathbf{L}_{EB} = \begin{bmatrix} \cos\theta\cos\psi & \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi \\ \cos\theta\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi \\ -\sin\theta & \sin\phi\cos\theta & \cos\phi\cos\theta \end{bmatrix} \tag{Eq. 4.24}$$

4.4.2. Orientation of the Airplane

Let the airplane experiences, in time Δt, an infinitesimal rotation from the position defined by Ψ, Θ, Φ to that corresponding to (Ψ+ΔΨ), (Θ+ΔΘ), (Φ+ΔΦ). The vector representing this rotation is approximately:

$$\Delta\mathbf{n} \approx \mathbf{i}_3\Delta\Phi + \mathbf{j}_2\Delta\Theta + \mathbf{k}_1\Delta\Psi \tag{Eq. 4.25}$$

Considering this rotation at Δt →0, and recalling that $\boldsymbol{\omega}_B = \begin{bmatrix} p & q & r \end{bmatrix}^T$, we get a set of differential equations in Eq. 4.26 from which the Euler angles can be calculated.

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \mathbf{T} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \quad \text{(Eq. 4.26)}$$

where,

$$\mathbf{T} = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{bmatrix} \quad \text{(Eq. 4.27)}$$

4.5. Euler's Equations of Motion

The set of equations in Eq. 4.16 summarize the motion of the airplane. In particular, the second equation shows that it is necessary the time derivative of the angular momentum, **h**, which contains the moments and products of inertia with respect to whatever axes are chosen.

By this, we come to the importance of the body axes. The inertias will be variables in the equations if the reference axes are fixed relative to the inertial space. Avoiding and simplifying futures mathematic operations, the motion equations are written relative to the frame F_B, in which all the inertias are constant.

So that, from Eq. 4.16 the force equation is:

$$\mathbf{f}_E = m\dot{\mathbf{V}}_E^E \quad \text{(Eq. 4.28)}$$

Expressing both, force and velocity, in the F_B:

$$\mathbf{L}_{EB}\mathbf{f}_B = m\frac{d}{dt}\left(\mathbf{L}_{EB}\mathbf{V}_B^E\right) = m\left(\dot{\mathbf{L}}_{EB}\,\mathbf{V}_B^E + \mathbf{L}_{EB}\,\dot{\mathbf{V}}_B^E\right) \qquad \text{(Eq. 4.29)}$$

In the Appendix A.4 of *[1]*, it is shown that the transformation matrix is:

$$\dot{\mathbf{L}}_{EB} = \mathbf{L}_{EB}\,\tilde{\omega}_B \qquad \text{(Eq. 4.30)}$$

Substituting Eq. 4.30 in Eq. 4.29 and pre multiplying it by $\mathbf{L}_{BE}$, we get

$$\mathbf{f}_B = m\left(\dot{\mathbf{V}}_B^E + \tilde{\omega}_B\,\mathbf{V}_B^E\right) \qquad \text{(Eq. 4.31)}$$

Remark: it is used Eq. 4.19 for $\tilde{\omega}_B$ and $\mathbf{V}_B^E = \begin{bmatrix} u^E & v^E & w^E \end{bmatrix}^T$.

Also, $\mathbf{f}_B$ is the sum of the aerodynamic force **A** plus the gravitational force $m\mathbf{g}$. That is,

$$\mathbf{f} = m\mathbf{g} + \mathbf{A} \qquad \text{(Eq. 4.32)}$$

where,

$$\mathbf{A} = \begin{bmatrix} X & Y & Z \end{bmatrix}^T \quad \text{and} \quad m\mathbf{g}_B = m\mathbf{L}_{EB}\mathbf{g}_E = m\mathbf{L}_{EB}\begin{bmatrix} 0 & 0 & g \end{bmatrix}^T$$

A similar process is applied into the moment equation of Eq. 4.16.

$$\mathbf{G}_B = \dot{\mathbf{h}}_B + \tilde{\omega}_B\,\mathbf{h}_B \qquad \text{(Eq. 4.33)}$$

where,

$\mathbf{G}_B = \begin{bmatrix} L & M & N \end{bmatrix}^T$ - it is the external moments of the airplane;

Assumption:

- The airplane is consider as a rigid body thus $\dot{\mathbf{I}}_B = 0$;

4.5.1. Choice of Body Axes

As it was discussed earlier in this section, all of these equations are valid for any orthogonal axes fixed in the airplane, with the origin at the center mass (point C, see Fig. 4.3) knew as *body axes*.

Because the most aircrafts are very nearly to the symmetrical geometry, it is usually to assume the exact symmetry, and then let to be C_{xz} the plane of symmetry. Consequently, the two products of inertia I_{xy} and I_{yz} are zero and simplifications can be made in Eq. 4.33.

Also, beyond these changes, it is considered that C_x points forward, C_z downward, C_y to the right and perpendicular to the plane of symmetry.

However, except to C_y axe, the others can have any direction. Conventionally, their directions are fixed in one of the three possibilities (see Fig. 4.3), discussed below:

- *Principal axes*: they coincide with the principal axes of the vehicle, so that all the products of inertia vanishes and remains only the moments of inertia;

- *Stability axes*: in these axes configuration, the C_x is aligned with **V** in a reference condition of steady symmetric flight and the reference values of v and w are zero (Lift force = weight of the airplane).

  These axes are chosen to build the following equations of this report because it simplifies them and, also, the expressions of the aerodynamic forces.

  However, it should note that for different initial flight conditions these axes are differently orientated in the airplane. Consequently, the values of I_x, I_y and I_{zx} will vary for each problem.

  Considering an angle ε between the stability axes (x_S) and principal axes (x_p), as shown the Fig. 4.3, the moments and product of inertia can be easily computed through the following equations:

$$I_x = I_{xp}\cos^2\varepsilon + I_{zp}\sin^2\varepsilon$$
$$I_z = I_{xp}\sin^2\varepsilon + I_{zp}\cos^2\varepsilon \qquad \text{(Eq. 4.34)}$$
$$I_{zx} = \frac{1}{2}\left(I_{xp} - I_{zp}\right)\sin 2\varepsilon$$

where,

I_{xp} and I_{zp} are the principal moments of inertia;

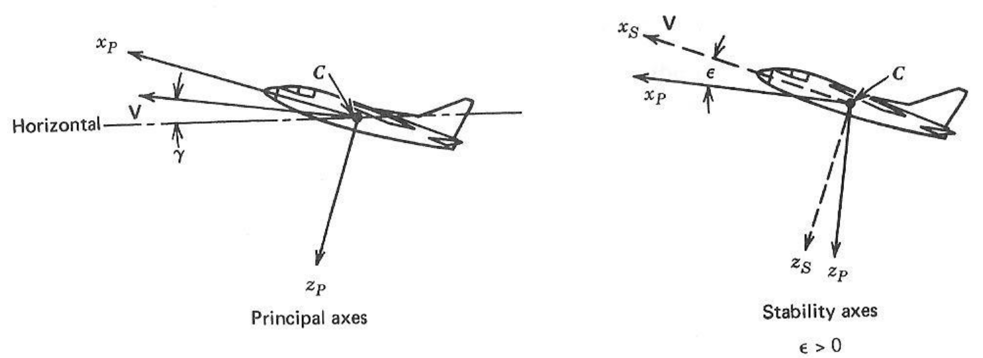

Fig. 4.3. Principal and Stability Axes

- *Body axes*: it is called any other configuration of axes that are neither the principal axes nor the stability axes.

4.6. The Equations Collected

The developed kinematical and dynamical equations in the foregoing sections are collected and they are listed below.

- *Force equations in F_B*:

$$X - mg\sin\theta = m(\dot{u}^E + qw^E - rv^E) \qquad (a)$$

$$Y + mg\cos\theta\sin\phi = m(\dot{v}^E + ru^E - pw^E) \qquad (b) \qquad \text{(Eq. 4.35)}$$

$$Z + mg\cos\theta\cos\phi = m(\dot{w}^E + pv^E - qu^E) \qquad (c)$$

- *Moments equations in F_B:*

$$L = I_x\dot{p} - I_{zx}\dot{r} + qr(I_z - I_y) - I_{zx}pq \qquad (a)$$

$$M = I_y\dot{q} + rp(I_x - I_z) + I_{zx}(p^2 - r^2) \qquad (b) \qquad \text{(Eq. 4.36)}$$

$$N = I_z\dot{r} - I_{zx}\dot{p} + pq(I_y - I_x) + I_{zx}qr \qquad (c)$$

- *Rotational kinematics equations:*

$$p = \dot{\phi} - \dot{\psi}\sin\theta \qquad (a)$$

$$q = \dot{\theta}\cos\phi + \dot{\psi}\cos\theta\sin\phi \qquad (b)$$

$$r = \dot{\psi}\cos\theta\cos\phi - \dot{\theta}\sin\phi \qquad (c)$$

$$\dot{\phi} = p + (q\sin\theta + r\cos\phi)\tan\theta \qquad (d) \qquad \text{(Eq. 4.37)}$$

$$\dot{\theta} = q\cos\phi - r\sin\phi \qquad (e)$$

$$\dot{\psi} = (q\sin\phi + r\cos\phi)\sec\theta \qquad (f)$$

- *Linear Kinematics equations:*

$$\begin{aligned}
\dot{x}_E &= u^E \cos\theta\cos\psi + v^E(\sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi) \\
&\quad + w^E(\cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi) && (a) \\
\dot{y}_E &= u^E \cos\theta\sin\psi + v^E(\sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi) \\
&\quad + w^E(\cos\phi\sin\theta\sin\psi - \sin\phi\sin\psi) && (b) \\
\dot{z}_E &= -u^E \sin t + v^E \sin\phi\sin\theta + w^E \cos\phi\cos\theta && (c) \\
u^E &= u + W_x && (d) \\
v^E &= v + W_y && (e) \\
w^E &= w + W_z && (f)
\end{aligned} \qquad \text{(Eq. 4.38)}$$

Although the Tab. 4.1, we reminding that:

Tab. 4.1. Brief description of the variables

X, Y, Z	Aerodynamic forces
L, M, N	Aerodynamic moments
u, v, w	Linear velocities relative to the air
p, q, r	Angular velocities relative to the air
x_E, y_E, z_E	Position of the C.G. in F_E
ψ, θ, ϕ	Aircraft attitude

The above equations consist of 15 coupled nonlinear ordinary differential equations in the independent variable *t*, and three algebraic equations (Eq. 4.38-*d..f)*. Also, they are quite general and contain a small number of assumptions. These are:

1. The airplane is a rigid body, which may have attached to it any number of rigid spinning rotors;

2. C_{xz} is a plane of mirror symmetry;

3. The effects of spinning rotors are negligible. This is the case when the airplane is in gliding flight with power off, when the symmetrical engines have opposite rotation, or when the rotor angular momentum is small.

It is clear that the aerodynamics forces and moments depend in some manner on the relative motion of the airplane with respect to the air (linear and angular velocities **V** and ***w***, respectively), on the control variables that fix the angles of any moveable surface and on the settings of any propulsion controls that determine the thrust vector.

For this reason, it is assumed in flight dynamics that the six forces and moments are functions of the six linear and angular velocities and of a *control vector,* represented by the following vector ***c***:

$$\mathbf{c} = \begin{bmatrix} \delta_a & \delta_e & \delta_r & \delta_p \end{bmatrix}^T$$

where,

the first three elements are the aileron, elevator and rudder angles and the last one is the throttle control;

4.7. Linearization of the Motion Equations

One of the interests in this present report is to study the motion of an aircraft in steady flight condition. Thus, it is assumed that the motion of the airplane consist of small deviations from one reference. By that, the equations of motions are frequently linearized by using the small-disturbance theory because of these reasons that follow below:

- satisfactory precision of unaccelerated flight;
- in many cases, the major aerodynamic effects are nearly linear functions of the disturbances;
- disturbed flight of considerable violence can occur with quite small values of the linear and angular-velocity disturbances;
- sufficient accuracy for engineering purposes to the response calculations where the disturbances are not infinitesimal;

However, some limitations are present in this theory. It is not suitable solving problems where there are large disturbance angles (example: $\Phi = \pi / 2$).

4.7.1. Notation for Small Disturbances

The reference flight conditions is assumed to be a symmetric flight and with no angular velocity. Thus, the initial reference values (denoted by a subscript zero) of the following variables is zero:

$$v_0 = p_0 = q_0 = r_0 = \phi_0 = \psi_0 = 0$$

Furthermore,

- As stability axes were selected as standard, $w_0 = 0$;
- By instant, it is assumed that wind velocity is zero, thus u_0 is equal to the reference flight speed;
- θ_0 is the initial angle of climb.

Beyond the notation of the initial reference values denoted by a subscript zero, the small perturbations are by prefix Δ. For convenience, when the initial value is zero, the Δ may be omitted.

Appling the small-disturbance notation in the equation of Sec. 4.6, and assuming that only the first-order terms in disturbance quantities are kept, then the following linear equations are obtained.

$$\begin{aligned} &X_0 + \Delta X - mg(\sin\theta_0 + \Delta\theta\cos\theta_0) = m\Delta\dot{u} && (a) \\ &Y_0 + \Delta Y + mg\phi\cos\theta_0 = m(\dot{v} + u_0 r) && (b) \\ &Z_0 + \Delta Z - mg(\cos\theta_0 - \Delta\theta\sin\theta_0) = m(\dot{w} - u_0 q) && (c) \end{aligned} \qquad \text{(Eq. 4.39)}$$

$$\begin{aligned} &L_0 + \Delta L = I_x\dot{p} - I_{zx}\dot{r} && (a) \\ &M_0 + \Delta M = I_y\dot{q} && (b) \\ &N_0 + \Delta N = -I_{zx}\dot{p} + I_z\dot{r} && (c) \end{aligned} \qquad \text{(Eq. 4.40)}$$

$$\begin{aligned} &\Delta\dot{\theta} = q && && (a) \\ &\dot{\phi} = p + r\tan\theta_0 && p = \dot{\phi} - \dot{\psi}\sin\theta_0 && (b) \\ &\dot{\psi} = r\sec\theta_0 && && (c) \end{aligned} \qquad \text{(Eq. 4.41)}$$

$$\dot{x}_E = (u_0 + \Delta u)\cos\theta_0 - u_0 \Delta\theta \sin\theta_0 + w\sin\theta_0 \qquad (a)$$
$$\dot{y}_E = u_0 \psi \cos\theta_0 + v \qquad (b) \qquad \text{(Eq. 4.42)}$$
$$\dot{z}_E = -(u_0 + \Delta u)\sin\theta_0 - u_0 \Delta\theta \cos\theta_0 + w\cos\theta_0 \qquad (c)$$

All the initial reference forces and moments are obtained when the disturbance quantities are zero from the above equations. Later, it is replaced these values on the above equations and is considered the disturbances quantities different from zero. By other words, it is translated the point of view in which these equations are analyzed: from the origin of the coordinate system to the *reference steady-state*.

4.7.2. Linear Air Reactions

What distinguishes from the other branches of mechanics in this problem is an aerodynamic ingredient that it contains. In atmospheric flight mechanics, exist one problem of determining and describing the aerodynamic forces and moments that act on a given body in arbitrary motion.

In the example described in *[1]*, "because the wings leaves behind it a vortex wake that in general generates an induced velocity field at the wing, and because hysteresis is present in flow separation processes, the aerodynamic field that fixes the lift at any given moment is actually dependent not only on the instantaneous value of *α* but strictly speaking on its entire past history".

Due to [3], the classical assumption of linear aerodynamic theory results that:

$$\Delta L(t_0) = L_\alpha \Delta\alpha + L_{\dot{\alpha}} \Delta\dot{\alpha} + L_{\ddot{\alpha}} \Delta\ddot{\alpha} + \ldots \qquad \text{(Eq. 4.43)}$$

where,

ΔL is the lift variation given by changes in the angle of attack (α) at the time;

Derivatives such as L_α in Eq. 4.43 are known as the *stability derivative,* or more generally as *aerodynamic derivatives*. They are calculated considering the reference conditions given at time t_0.

For a truly symmetric configuration, as it is considered in this case, we noted that the side force *Y*, the rolling moment *L*, and the yawing moment *N* will all be exactly zero in any condition of symmetric flight. By other words, $\beta, p, r, \phi, \psi, \text{and } y_E$ are all identically zero.

For writing out the complete linear expression for the aerodynamic force and moments of interest (dynamic longitudinal stability), the fact described above is considered and, in addition, we make the further approximations:

1 – We may neglect as well all the derivatives of the symmetric forces and moments with respect to the asymmetric motion variables;

2 – We may neglect all derivatives with respect to rates of change of motion variables except for $Z_{\dot{w}} \text{ and } M_{\dot{w}}$;

3 – The derivative X_q is also negligibly small;

4 – The density of the atmosphere is assumed not to vary with altitude;

With these assumptions, the linear forces and moments that influence the longitudinal motion are:

$$\Delta X = X_u \Delta u + X_w w + \Delta X_c \qquad (a)$$
$$\Delta Z = Z_u \Delta u + Z_w w + Z_{\dot{w}} \dot{w} + Z_q q + \Delta Z_c \qquad (b) \qquad \text{(Eq. 4.44)}$$
$$\Delta M = M_u \Delta u + M_w w + M_{\dot{w}} \dot{w} + M_q q + \Delta M_c \qquad (c)$$

Reminding that the terms on the right with subscript *c* are control forces and moments that result from the control vector **c**.

Substituting (Eq. 4.44*a* to *c*) in (Eq. 4.39*a,*c and Eq. 4.40*b*) respectively, and rearranging them in additional with (Eq. 4.41*a*), (Eq. 4.42*a* and *b*) is obtained the *Linearized Longitudinal Equations of Motion* showed in Eq. 4.45.

$$\begin{bmatrix} \Delta\dot{u} \\ \dot{w} \\ \dot{q} \\ \Delta\dot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{X_u}{m} & \frac{X_w}{m} & 0 & -g\cos\theta_0 \\ \frac{Z_u}{m-Z_{\dot{w}}} & \frac{Z_w}{m-Z_{\dot{w}}} & \frac{Z_q+mu_0}{m-Z_{\dot{w}}} & \frac{-mg\sin\theta_0}{m-Z_{\dot{w}}} \\ \frac{1}{I_y}\left[M_u+\frac{M_{\dot{w}}Z_u}{\left(m-Z_{\dot{w}}\right)}\right] & \frac{1}{I_y}\left[M_w+\frac{M_{\dot{w}}Z_w}{\left(m-Z_{\dot{w}}\right)}\right] & \frac{1}{I_y}\left[M_q+\frac{M_{\dot{w}}\left(Z_q+mu_0\right)}{\left(m-Z_{\dot{w}}\right)}\right] & -\frac{M_{\dot{w}}mg\sin\theta_0}{I_y\left(m-Z_{\dot{w}}\right)} \\ 0 & 0 & 1 & 0 \end{bmatrix} \times$$

$$\begin{bmatrix} \Delta u \\ w \\ q \\ \Delta\theta \end{bmatrix} + \begin{bmatrix} \frac{\Delta X_c}{m} \\ \frac{\Delta Z_c}{m-Z_{\dot{w}}} \\ \frac{\Delta M_c}{I_y}+\frac{M_{\dot{w}}}{I_y}\frac{\Delta Z_c}{\left(m-Z_{\dot{w}}\right)} \\ 0 \end{bmatrix}$$

$$\Delta\dot{x}_E = \Delta u\cos\theta_0 + w\sin\theta_0 - u_0\Delta\theta\sin\theta_0$$
$$\Delta\dot{z}_E = -\Delta u\sin\theta_0 + w\cos\theta_0 - u_0\Delta\theta\cos\theta_0$$

Longitudinal Equations, Eq. 4.45

Remark: in the following development of the theory part, the matrix equation is considered to be as:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\Delta\mathbf{c}$$

4.8. Effects of Wind

Intended to simulate a real case and kwon that many airplanes accidents have been attributed to phenomena of air turbulence, for example, at the presence of downbursts phenomena; it is important and necessary to examine the effects of nonuniform and unsteady motion of the atmosphere on the behavior of flight vehicles.

However, because of the limitation imposed by the linear model based on small disturbances from a steady reference condition, is considered that, quite apart from the turbulence, the wind have a mean structure which is not uniform in space.

One relevant steady state, which is allowed to be investigated with the developed linear model, is the horizontal flight in the boundary layer. The planetary boundary layer has characteristics quite similar to the classical flat-plate turbulent boundary layer of aerodynamics. The vertical extent of this layer in strong winds depends mainly on the roughness of the underlying terrain. Fig. 4.4 shows the power-law profiles associated with different roughness.

Considering that, around the reference flight altitude, the wind is specified to vary linearly with the altitude with gradient $\Gamma = dW/dz_E$ and to be parallel to the *xz* plane. The wind vector is prescribed in frame F_E as

$$\mathbf{W}_E = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \left(W_0 + \Gamma z_E \right) \qquad \text{(Eq. 4.46)}$$

where,

W_0 is the strength of a tailwind at the reference height;

$\Gamma = -\mathrm{sgn}(W_0)\left|\Gamma\right|$ for wind that increases with altitude;

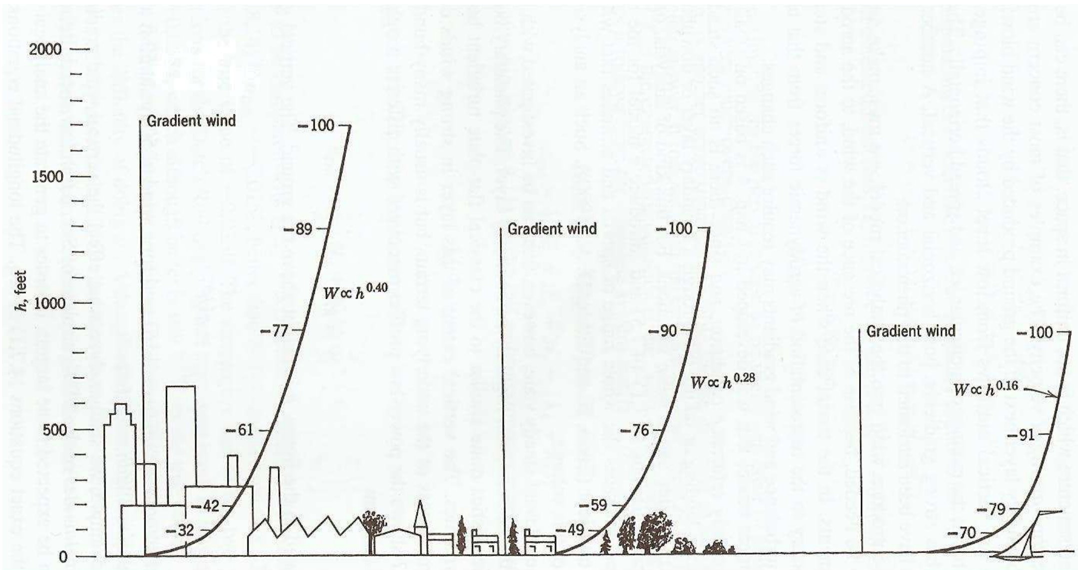

Fig. 4.4. Profiles of Mean Wind Velocity over Level Terrains of Different Roughness *[1].*

To represent $\mathbf{W}_E$ on the body axes, it is used the transformation matrix $\mathbf{L}_{BE}$ (transpose of Eq. 4.24). It is considered also that small angles occurs and $\phi = \psi = 0$. In addition, it is neglected the second-order product $z_E\theta$ to get

$$W_B = \begin{bmatrix} W_0 + \Gamma z_E \\ 0 \\ (W_0 + \Gamma z_E)\theta \end{bmatrix} \approx \begin{bmatrix} W_0 + \Gamma z_E \\ 0 \\ W_0\theta \end{bmatrix} \qquad \text{(Eq. 4.47)}$$

Similarly to get above, Eq. 4.24 is obtained replacing u^E and w^E through the knowing relation $\mathbf{V}^E = \mathbf{V} + \mathbf{W}$, and later, eliminating $\dot{\theta}$ and $\dot{z}_E$ with Eq. 4.41*a* and Eq. 4.42*c*. Thus,

$$\mathbf{A} = \begin{bmatrix} \dfrac{X_u}{m} & \left(\dfrac{X_w}{m} - \Gamma\right) & 0 & \left(-g\cos\theta_0 + \Gamma u_0\right) \\ & & \mathbf{A}' & \end{bmatrix} \qquad \text{(Eq. 4.48)}$$

A is the new matrix (4x4) of Eq. 4.45 which it takes account the influence of the wind in the aircraft motion, and **A**' is the last three rows of the matrix of Eq. 4.45.

4.9. Nondimensional System

There are many reasons to indicate the great advantage of using nondimensional coefficients for aerodynamic forces and moments such as lift, drag, and pitching moment. Those important ones are:

- In this way the major effects of speed, size, and air density are automatically accounted for;
- Nondimensional coefficients are needed for the many derivatives - X_u and so on – that occur in above. Unfortunately, there is no universally accepted standard for these coefficients, although attempts have been made to devise one (e.g., ANSI/AIAA, 1992 in ref. [4]);

Digressing briefly to a dimensional analysis of the general flight dynamics problem, imagine a class of geometrically similar airplanes of various sizes and masses in steady unaccelerated flight at various heights and speeds. It is supposed now, that one of these airplanes is subjected to a disturbance, and, by consequence, some typical nondimensional variable Π (e.g., pitching angle, the load factor, or the helix angle in roll) varies with time. Thus, Π = *f*(t).

Let it to be assumed that this equation can be generalized to cover the whole class of airplanes, under all flight conditions. Then, not only as a variable of time, we can write it as:

$$\Pi = f(u_0, \rho, m, l, g, M, RN, t) \qquad \text{(Eq. 4.49)}$$

where,

m, l : these are the airplane mass and the its characteristic length;

M : Mach number;

RN: Reynolds number;

Buckingham's Π theorem (in ref. [5]) tell that, since there are eight quantities in Eq. 4.49 containing three fundamental dimensions, L, M and T, then there are 8 – 3 = 5 independent dimensionless combinations of eight quantities.

Following the Π theorem, Eq. 4.49 becomes:

$$\Pi = f(M, RN, FN, \mu, t/t^*) \qquad \text{(Eq. 4.50)}$$

where,

$$\text{Froude Number:} \qquad FN = \frac{u_0^2}{gl}$$

$$\text{Relative density parameter:} \qquad \mu = \frac{m}{\rho S l}$$

$$t^* = \frac{l}{u_0}$$

4.9.1. Nondimensional Stability Derivatives

The nondimensional stability derivatives are the partial derivatives of the forces and moment coefficients (in lines 1, 3 and 4 of Tab. 4.2) with respect to the nondimensional motion variables (in lines 5, 6 and 7).

Tab. 4.2. - The Nondimensional System

Dimensional Quantity	Divisor		Nondimensional Quantity
	General Case	Small Disturbance Case	
X,Y,Z	$\frac{1}{2}\rho V^2 S$	$\frac{1}{2}\rho V^2 S$	C_x, C_y, C_z
mg	$\frac{1}{2}\rho V^2 S$	$\frac{1}{2}\rho V^2 S$	C_w
M	$\frac{1}{2}\rho V^2 S\bar{c}$	$\frac{1}{2}\rho V^2 S\bar{c}$	C_m
L,N	$\frac{1}{2}\rho V^2 Sb$	$\frac{1}{2}\rho V^2 Sb$	C_l, C_n
u,v,w	V	u_0	$\hat{u}, \hat{v}, \hat{w}$
$\dot{\alpha}, q$	$2V/\bar{c}$	$2u_0/\bar{c}$	$\hat{\dot{\alpha}}, \hat{q}$
$\dot{\beta}, p, r$	$2V/b$	$2u_0/b$	$\hat{\dot{\beta}}, \hat{p}, \hat{r}$
m	$\rho S\bar{c}/2$	$\rho S\bar{c}/2$	μ
I_y	$\rho S(\bar{c}/2)^3$	$\rho S(\bar{c}/2)^3$	$\hat{I}_y$
I_x, I_z, I_{zx}	$\rho S(b/2)^3$	$\rho S(b/2)^3$	$\hat{I}_x, \hat{I}_z, \hat{I}_{zx}$
T	--	$t^* = \bar{c}/(2u_0)$	$\hat{t}$

The notation for the longitudinal derivative coefficients is displayed in the Tab. 4.3.

Tab. 4.3. Longitudinal Nondimensional Derivatives

	C_x	C_z	C_m
$\hat{u}$	C_{x_u}	C_{z_u}	C_{m_u}
α	C_{x_α}	C_{z_α}	C_{m_α}
$\hat{q}$	C_{x_q}	C_{z_q}	C_{m_q}
$\hat{\dot{\alpha}}$	$C_{x_{\hat{\dot{\alpha}}}}$	$C_{z_{\hat{\dot{\alpha}}}}$	$C_{m_{\hat{\dot{\alpha}}}}$
δ_e	$C_{x_{\delta_e}}$	$C_{z_{\delta_e}}$	$C_{m_{\delta_e}}$

The variable δ_e is the parameter that represents the angle variation of the elevator's surface which it has an important influence in the longitudinal motion of the airplane.

4.10. Dimensional Stability Derivatives

To solve Eq. 4.45, we need expressions for the dimensional derivatives in terms of the nondimensional derivatives. A few examples of these are derived as follows to illustrate the procedure utilized, and the whole set need is displayed in the Tab. 4.4.

4.10.1. The Z Derivatives

From Tab. 4.2, $Z = 1/2 C_z \rho V^2 S$ where $V^2 = u^2 + v^2 + w^2$ and $u = u_0 + \Delta u$. Considering the subscript zero to indicate the reference flight condition, we have:

$$Z_u = \left(\frac{\partial Z}{\partial u}\right)_0 = C_{z_0} \rho u_0 S \left(\frac{\partial V}{\partial u}\right)_0 + \frac{1}{2} \rho u_0^2 S \left(\frac{\partial C_z}{\partial u}\right)_0 \qquad \text{(Eq. 4.51)}$$

Also,

$$\left(\frac{\partial C_z}{\partial u}\right)_0 = \frac{1}{u_0}\left(\frac{\partial C_z}{\partial \hat{u}}\right)_0 = \frac{1}{u_0} C_{z_u}$$

From Eq. 4.39*c*, considering the disturbances equal zero, we get one of the reference steady state:

$$Z_0 = -mg\cos\theta_0 \quad \Leftrightarrow \quad C_{z_0} = -C_{w_0}\cos\theta_0$$

So that,

$$Z_u = -\rho u_0 S C_{w_0}\cos\theta_0 + \frac{1}{2}\rho u_0 S C_{z_u}$$

Also, since $(\partial V / \partial w)_0 = 0$ then

$$Z_w = \left(\frac{\partial Z}{\partial w}\right)_0 = \frac{1}{2}\rho u_0^2 S\left(\frac{\partial C_z}{\partial w}\right)_0$$

But $w = u_0\alpha_x$, so that

$$Z_w = \frac{1}{2}\rho u_0 S\left(\frac{\partial C_z}{\partial \alpha_x}\right)_0 = \frac{1}{2}\rho u_0 S C_{z_\alpha}$$

In a similar way,

$$Z_{\dot{w}} = \left(\frac{\partial Z}{\partial \dot{w}}\right)_0 = \frac{1}{2}\rho u_0^2 S\left(\frac{\partial C_z}{\partial \dot{w}}\right)_0 = \frac{1}{2}\rho u_0 S\left(\frac{\partial C_z}{\partial \dot{\alpha}_x}\right)_0$$

But

$$\left(\frac{\partial C_z}{\partial \dot{\alpha}_x}\right)_0 = \frac{\bar{c}}{2u_0} C_{z_{\dot{\alpha}}}$$

Hence

$$Z_{\dot{w}} = \frac{1}{4}\rho \bar{c} S C_{z_{\dot{\alpha}}}$$

Also,

$$Z_q = \left(\frac{\partial Z}{\partial q}\right)_0 = \frac{1}{2}\rho u_0^2 S\left(\frac{\partial C_z}{\partial q}\right)_0$$

But

$$q = 2\frac{u_0}{\overline{c}}\hat{q}$$

Hence

$$Z_q = \frac{1}{4}\rho u_0 S\overline{c}C_{z_q}$$

4.10.2. The X Derivatives

These are found in a manner similar to the Z derivatives. In this instance from Eq. 4.39*a*, considering the disturbances equal zero, then

$$C_{x_0} = C_{w_0}\sin\theta_0$$

4.10.3. The M Derivatives

These are also found in a manner similar to the Z derivatives. In this case we start with $M = C_m \frac{1}{2}\rho V^2 S\overline{c}$ and note from Eq. 4.40*b* the reference steady state for M is equal zero. Thus, $C_{m_0} = 0$.

Tab. 4.4. Longitudinal Dimensional Derivatives

	X	Z	M
u	$\rho u_0 S C_{w_0} \sin\theta_0 + \frac{1}{2}\rho u_0 S C_{x_u}$	$-\rho u_0 S C_{w_0} \cos\theta_0 + \frac{1}{2}\rho u_0 S C_{z_u}$	$\frac{1}{2}\rho u_0 \bar{c} S C_{m_u}$
w	$\frac{1}{2}\rho u_0 S C_{x_\alpha}$	$\frac{1}{2}\rho u_0 S C_{z_\alpha}$	$\frac{1}{2}\rho u_0 \bar{c} S C_{m_\alpha}$
q	$\frac{1}{4}\rho u_0 \bar{c} S C_{x_q}$	$\frac{1}{4}\rho u_0 \bar{c} S C_{z_q}$	$\frac{1}{4}\rho u_0 \bar{c}^2 S C_{m_q}$
$\dot{w}$	$\frac{1}{4}\rho \bar{c} S C_{x_{\dot{\alpha}}}$	$\frac{1}{4}\rho \bar{c} S C_{z_{\dot{\alpha}}}$	$\frac{1}{4}\rho \bar{c}^2 S C_{m_{\dot{\alpha}}}$

4.10.4. The Control Derivatives

On the right side of Eq. 4.45, we have the product $\mathbf{B}\Delta\mathbf{c}$ which $\mathbf{c}$ corresponds to the control vector. For the longitudinal control, it is assumed that the available controls are well enough represented by

$$\mathbf{c} = \begin{bmatrix} \delta_e & \delta_p \end{bmatrix} \qquad \text{(Eq. 4.52)}$$

and the incremental aerodynamics forces and moment that result from their actuation are given by a set of *control derivatives* in the form

$$\begin{bmatrix} \Delta X_c \\ \Delta Z_c \\ \Delta M_c \end{bmatrix} = \begin{bmatrix} X_{\delta e} & X_{\delta p} \\ Z_{\delta e} & Z_{\delta p} \\ M_{\delta e} & M_{\delta p} \end{bmatrix} \begin{bmatrix} \Delta\delta_e \\ \Delta\delta_p \end{bmatrix} \qquad \text{(Eq. 4.53)}$$

which the dimensional derivatives are calculated as

$$
\begin{aligned}
X_{\delta e} &= C_{x\delta e}\,\tfrac{1}{2}\rho u_0^2 S\\
Z_{\delta e} &= C_{z\delta e}\,\tfrac{1}{2}\rho u_0^2 S\\
M_{\delta e} &= C_{m_{\delta e}}\,\tfrac{1}{2}\rho u_0^2 S\bar{c}
\end{aligned}
\qquad \text{(Eq. 4.54)}
$$

$C_{x\delta e}, C_{z\delta e}, C_{m\delta e}$ are the nondimensional stability derivatives;

Unlike to these ones, for the throttle it is arbitrarily choose a value of

$$
\begin{aligned}
X_{\delta p}/m &= 0.3g \xrightarrow{when} \delta_p = 1\\
Z_{\delta p} &= M_{\delta p} = 0
\end{aligned}
$$

The use of constant derivatives, as in Eq. 4.37, to describe the force output of the propulsion system in response to throttle input does not allow for any time lag in the build of engine thrust since it implies that the thrust is instantaneously proportional to the throttle position (δ_p). This is not unreasonable for propeller airplanes, but it is not a good model for jets in situations when the short-term response is important *[1].*

This effect is allowed in the time domain by adding additional differential equation and an additional variable; or in the Laplace domain by using control transfer functions instead of control derivatives.

Substituting Eq. 4.37 into Eq. 4.45 we derive the matrix **B** to be

$$
\mathbf{B} = \begin{bmatrix}
\dfrac{X_{\delta e}}{m} & \dfrac{X_{\delta p}}{m}\\
\dfrac{Z_{\delta e}}{\left(m - Z_{\dot w}\right)} & \dfrac{Z_{\delta p}}{\left(m - Z_{\dot w}\right)}\\
\dfrac{M_{\delta e}}{I_y} + \dfrac{M_{\dot w}}{I_y}\dfrac{Z_{\delta e}}{\left(m - Z_{\dot w}\right)} & \dfrac{M_{\delta p}}{I_y} + \dfrac{M_{\dot w}}{I_y}\dfrac{Z_{\delta p}}{\left(m - Z_{\dot w}\right)}\\
0 & 0
\end{bmatrix}
\qquad \text{(Eq. 4.55)}
$$

5. Experimental Work Development

5.1. Objectives

Interested in aeronautic field, more specifically applied in dynamics of the flight jointly with automatic control system, this present report is destined to simulate the longitudinal motion of one passenger aircraft (Boeing 747-100) commanded by an auto pilot control system on adverse atmosphere condition. It consists in analyzing flight parameters, for different initial flight and atmosphere conditions, during the transition movement from its actual flight trajectory to the ordered one (given by the aircrafts navigation system), until it get the steady-state condition.

In other words, the airplane is assumed to be in steady-state flight, parallel to the reference trajectory ($\theta_0 = 0$), and with the same planned velocity. However, its initial height is perturbed considering be over or under to the final one.

Closing to reality, the control system mentioned consists to one first simple approximate model of one auto-pilot (from ref. *[1]*), by using closed loop control method, usually present in the automatic controlled systems. This is purposed to correct the flight altitude and keep the flight speed by acting on the control parameters (i.e., elevator angle and propulsion level).

For the simulation, the model is based on the linear equations Eq. 4.45, in which represents its longitudinal motion and its control subsystem. It is displayed and developed in *Simulink/Matlab®* by using block-diagram representation, in the Laplace domain. In the following of this chapter, figures show the equivalence between equations and their block-diagram representation.

Initially, the Fig. 5.1 shows a conceptual idea developed for this model. It is subdivided in two big subsystems: control and airframe subsystems.

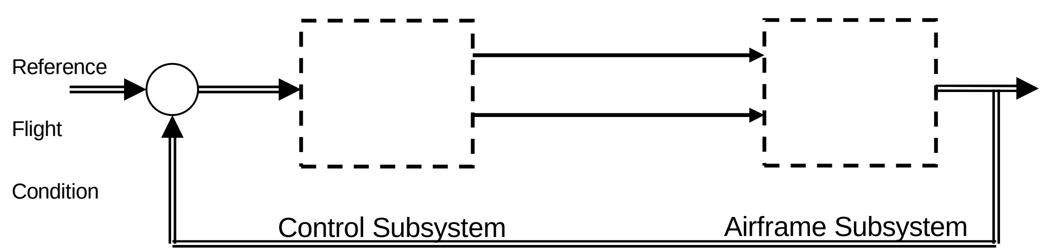

Fig. 5.1. Scheme of a General Model for Longitudinal Control of an Aircraft

The role of this chapter is to present, each subsystem expanded for this project with their values and details. Finally, it will be described the different initial flight and atmosphere conditions, in which this report is destined to study.

5.2. Description of the Boeing 747 - 100

The Boeing 747, commonly nicknamed the "Jumbo Jet", is a long-haul, wide body commercial airliner. Known for its impressive size, it is among the world's most recognizable aircraft [9].

The four-engine 747, produced by Boeing's Commercial Airplane unit, uses a double decker configuration for part of its length. A typical three-class layout accommodates 416 passengers, while a two-class layout accommodates a maximum of 524 passengers. The hump created by the upper deck has made the 747 a highly recognizable icon of air travel [8].

The model 100 had its first entered service in January 1970, and since then, it has continued to be developed through a series of models and special versions. As of May 1990, only versions of the model 400 were being marketed. By the year 1994, close to 800 Boeing-747s were in operation around the world, and the aircraft was still in production. *[1]*

A three view drawing of the aircraft is given in Fig. 5.2. A body axes system F_B is located with the origin at the CG and its *x*-axis along the fuselage reference line (FRL). The CG is located at $0.25\bar{c}$, and this is the location that applies for the following tabulated data.

One flight case, based on Heffley and Jewell (ref. [7]), is documented through data presented on Tab. 5.1. It represents straight and level steady-flight at a fixed-altitude with flaps retracted and the gears up.

It should be noted that the moments and product of inertia are given relative to the body frame F_B shown in the Fig. 5.2. Moreover, in the Table 5.2 is given the nondimensional stability derivatives.

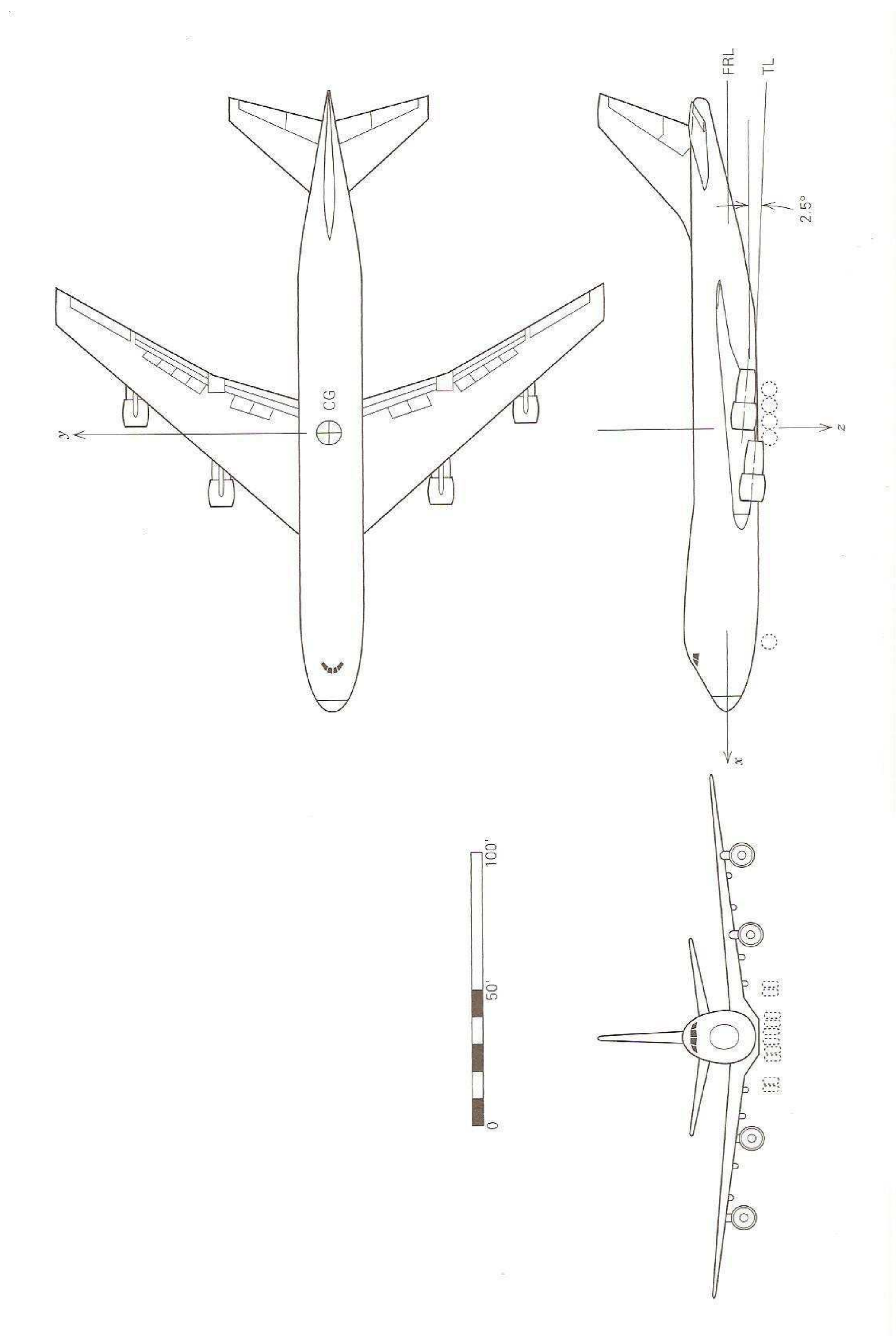

Fig. 5.2. Boeing 747 – 100. *[1]*

Tab. 5.1. Boeing 747-100 Data

	Case
Altitude (m)	12192
M	0.9
V(m/s)	265.48
W (kg)	2886.44×10^5
I_x (kg m^2)	2.46683×10^7
I_y (kg m^2)	4.48637×10^7
I_z (kg m^2)	6.73634×10^7
I_{zx} (kg m^2)	1.31474×10^6
ξ (degrees)	-2.4
C_D	0.043
Wing Area – S (m^2)	510.97
Wing Span – b (m)	59.64
Mean Aerodynamic Chord - $\bar{c}$ (m)	8.324
CG position, fraction of the Mean Chord – h	0.25
Lift Coefficient – C_L	0.654

Tab. 5.2. Nondimensional Longitudinal Derivatives

	C_x	C_z	C_m
$\hat{u}$	-0.1080	-0.1060	0.1043
α	0.2193	-4.920	-1.023
$\hat{q}$	0	-5.921	-23.92
$\hat{\dot{\alpha}}$	0	5.896	-6.314

5.3. Control Subsystem

Generally, this control subsystem, as seen in Fig. 5.3, is based on closed-loop control and is responsible to make the error equal to zero between its instantaneous, measured by sensors, and its reference value. The last one can be defined as a set point or can be calculated based on other variables which may vary in time.

Regarding the objectives for this longitudinal control problem, this subsystem is aimed to control the flight altitude (*h*) and the aircraft velocity (*u*), which becomes these variables, important in the analyses.

However, once that it is usually necessary to have one pitch angle (θ) to correct an altitude difference, it is evident that a secondary variable influences the result. Thus, the elevator angle (δ_e) is essential too.

The values h_0 and u_0 are input values of reference, in which characterize the ordered trajectory from the aircrafts navigation control system.

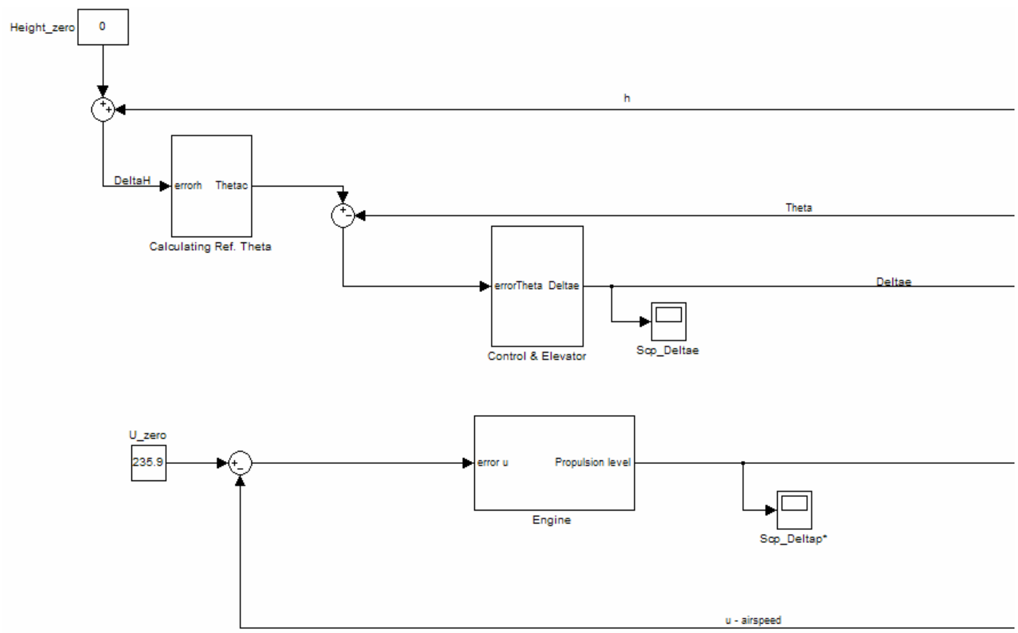

Fig. 5.3. General Schema of the Control Subsystem

5.3.1. Calculating Reference Pitch Angle

Any error, between the reference and its actual flight altitude, produces, by consequence, a reference pitch angle which determines the angle that the aircraft should follow to correct its trajectory. This inclination (or pitch angle) is calculated by multiplying the error with one constant of proportionality *K*, as is shown in the Fig. 5.4.

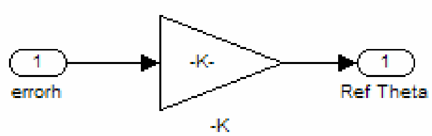

Fig. 5.4. Calculating Reference Pitch Angle Subsystem

The value of *K* is considered, according to *[1]*, as:

$$K = -6.56178 \times 10^{-4} \quad \left[{}^{\circ}\!/_{m}\right]$$

Its negative value is due considering that for any positive pitch angle, it is necessary a negative variation in altitude (*h*). Note that altitude *h* is the opposite of z_E used in the Chap. 4.

5.3.2. Control & Elevator

The command variable, which rights any error on the airplane inclination, is the elevator angle. It is calculated, as shown the Fig. 5.5, using a PID element of control. To make it more realistic is added one first order lag element in which is mainly associated with the servo actuator performing on the elevator.

In the Laplace domain, the transfer function J_e(s) relates the output and the input through equation in Eq. 5.1.

$$J_e(s) = \frac{\Delta\delta_e}{e_\theta} = \frac{(a_0 s^{-1} + a_1 + a_2 s\ )}{(1+\tau_e s)} \qquad \text{(Eq. 5.1)}$$

where,

a_0, a_1, a_2 are respectively the integral, proportional and derivative constants in the PID control element;

τ_e is the time constant due to the servo-actuator;

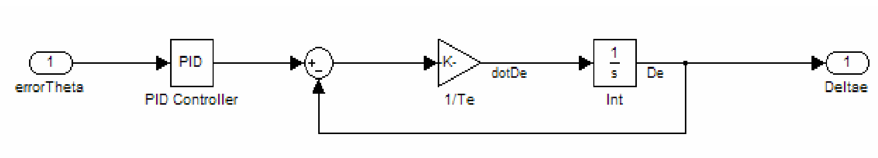

Fig. 5.5. Control & Elevator Subsystem

From *[1]*, the values of the constants in PID control element, and the time constant are:

$$a_0 = a_1 = a_2 = -0.5 \text{ and } \tau_e = 0.1s$$

5.3.3. Engine Control

Beyond to maintain the airplane velocity constant, the engine control subsystem allows changes on the throttle level of the turbo-engines, in which interferes indirectly, on the altitude variable, through the lift force. As a result of it, for one better altitude controller, it is necessary one engine control too.

To make the simulation more realistic, it is included a first-order lag element on the engine control. This is inherent in the build up of a thrust of a jet engine following sudden movements of the throttle.

Another feature, that is incorporated to add realism into the control, is a thrust limiter. Because of a transport aircraft inherently respond slowly to changes in thrust, the gains chosen, to give satisfactory response for very small perturbations in speed, lead to a demand in thrust may be outside of the engine capacity, which envelopes larger speed errors.

Therefore, it is included a nonlinear feature that limits the thrust to range $0 \leq T \leq 1.1T_0$. Remarking that thrust T_0 is equal to the drag force D_0, when the lift is kept exactly equal to the weight, at all time.

Such range, described above, corresponds to the cruising flight phase and it contains the implicit assumption (quite arbitrary) that the airplane, flying near its ceiling, has 10% additional thrust available, and that idling engines correspond to zero thrust. The Fig. 5.6 shows a general schema for the engine control subsystem.

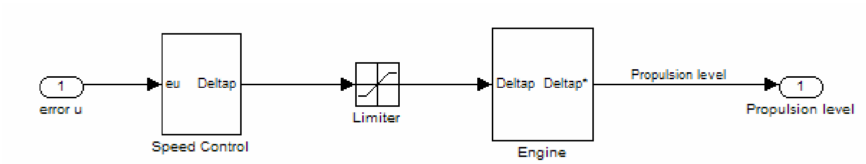

Fig. 5.6. General Schema for the Engine Control Subsystem

Also, the PID control element exists on the engine subsystem, represented in the Fig. 5.7 by the Speed Control element, and it is responsible to calculate the output (throttle level - δ_p) based on the error of the airplane speed.

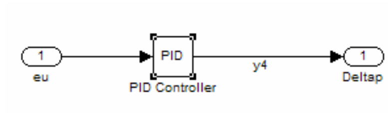

Fig. 5.7. Speed Control Element

The transfer function C_p, which describes the relation between the output δ_P and input e_u, is represented, in the Laplace domain, through equation (5.3,2).

$$C_p(s) = \frac{\Delta\delta_p}{e_u} = (b_0 s^{-1} + b_1 + b_2 s \ ) \qquad \text{(Eq. 5.2)}$$

where,

b_0, b_1, b_2 are respectively the integral, proportional and derivative constants in the PID control element;

According to *[1],* these constants have the followings values:

$$b_0 = 0.005 \quad b_1 = 0.08 \quad b_2 = 0.16$$

From the values of C_{D0} and $C_{L0,}$ given on Tab. 5.1, we find that $D_0 = T_0 = 0.0657W$. Moreover, in section 4.10, it is given that $\delta_p = 1$ corresponds to a thrust of 0.3W. To obtain the zero net thrust, it corresponds for the output - y_4 - of the PID element, as is shown in the Fig. 5.7, to be

$$y_4 = -0.0657 / 0.3 = -0.219$$

The nonlinear relationship of y_4 is, therefore, implemented in the computing program by a limiter element displayed in the Fig. 5.6, and equivalent to

$$\Delta\delta_p = y_4$$
$$IF\ y_4 < -0.219\ THEN\ \Delta\delta_p = -0.219$$
$$IF\ y_4 > 0.10\ THEN\ \Delta\delta_p = 0.10$$

where,

the maximum engine thrust has been assumed to be 10% greater than cruising thrust in high altitudes.

However, during the phase flight approximation (i.e., in landing process), it is characterizes with small flight altitudes, and the superior limit of the thrust engine is approximately around *1.5*T_0. Thus, $0 \le T \le 1.5T_0$.

Therefore, similar to cruising flight phase, the nonlinear relationship for y_4 in approximation flight phase, is given by

$$\Delta\delta_p = y_4$$
$$IF\ y_4 < -0.219\ THEN\ \Delta\delta_p = -0.219$$
$$IF\ y_4 > 0.50\ THEN\ \Delta\delta_p = 0.50$$

where,

the maximum engine thrust has been assumed to be 50% greater than cruising thrust for small altitudes.

Remarking that, y_4 is equivalent to $\Delta\delta_p$ in both cases.

Complementary, the engine element that appears in Fig. 5.6, in the Laplace domain, has the following transfer function

$$J_p(s) = \frac{\delta p^*}{\delta p} = \frac{1}{(1+\tau_p s)}$$

where,

τ_p is time constant and is equal to 3.5, according to *[1]*;

5.4. Airframe Subsystem

As its own name explains, this subsystem corresponds to aircraft and environmental models. It is responsible to calculate the answers, such as the flight path and aircraft variables (i.e., altitude, attitude, airspeed, etc.), derived from the inputs given by the control subsystem.

The Fig. 5.8 illustrates a schema of the airframe subsystem. As can be observed, it contains many others intern subsystems, which are dependent from each other. They are:

- Data : contains data about the aircraft and air ;
- Auxiliary: makes complementary calculations to facilitate the mathematical operation for others subsystems;
- Xdeltap, Mu_Mdotw_Mq, Xu_Xw_Zu_Zw, Xdeltae_Zdeltae_Mdeltae: calculate dimensional stability derivatives necessaries to be used on the motion equations;
- eq_ze, eq_q, eq_w, eq_u, eq_Theta: they calculate the motion variables through equations given in Eq. 4.45;
- WindData: contains data about the atmosphere wind and calculates the necessaries parameters to be added on the motion equations;

In the following of this section, details about each intern subsystems will be explored and presented.

Reminding that, all variables are expressed in S.I. (International Measurement System), and all assumptions, listed in the last and in begin of this chapter, are being used on the calculations.

5.4.1. Data Subsystem

As its own name explains, this subsystem contains all necessaries data to be used in the equations.

Generally, these data correspond to the airplane geometries, aerodynamic coefficients, air and physics properties. In Tab. 5.3 are included the relationship existent between their nicked name, used on the development of the program, with their nomenclature, used on the report.

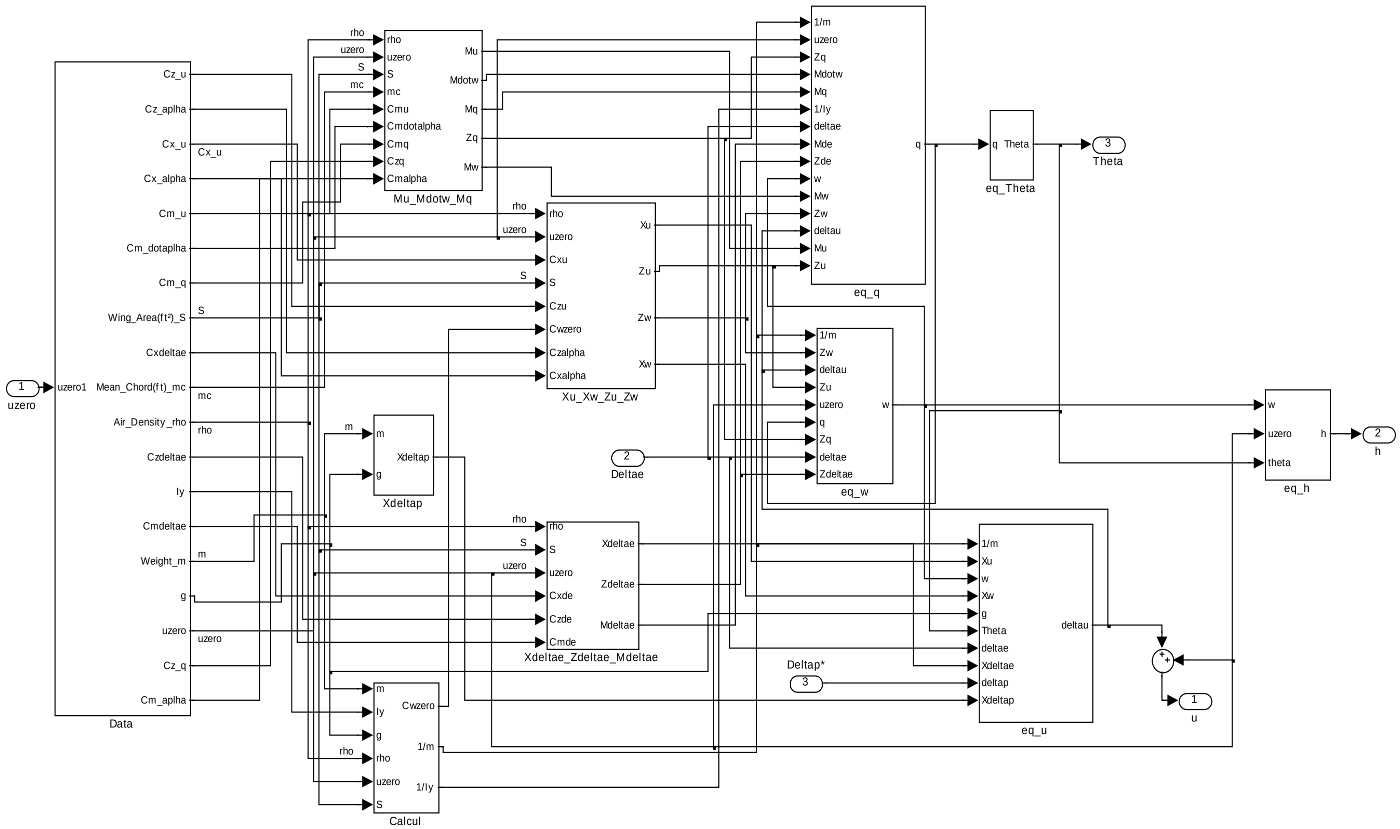

Fig. 5.8. Schema General of Airframe Subsystem

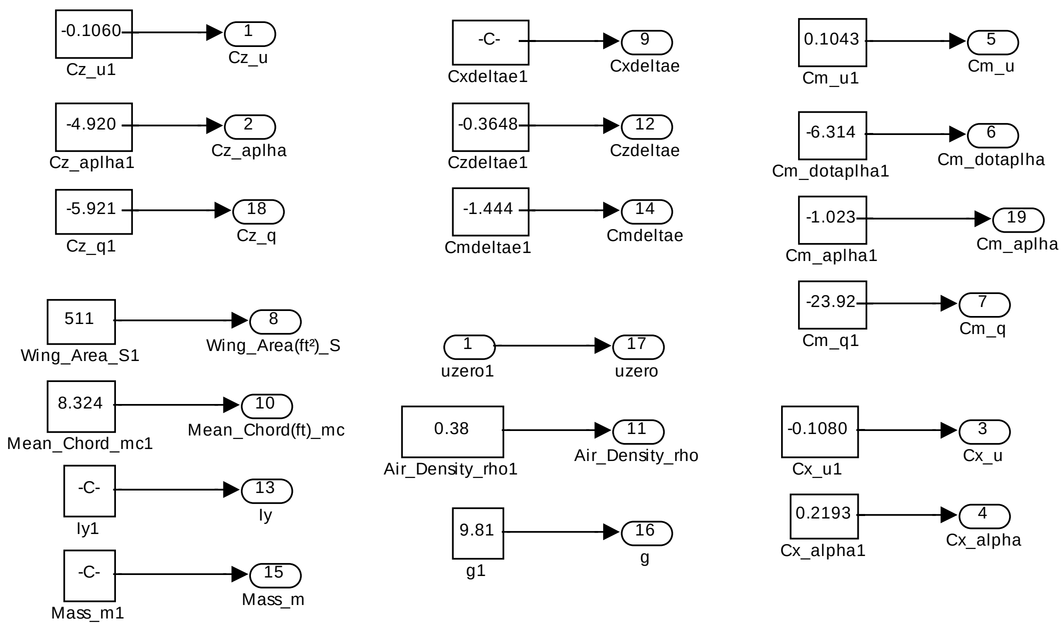

Fig. 5.9. Data Subsystem

The nondimensional stability derivatives are obtained from Tab. 5.2. The Fig. 5.9 illustrates this subsystem.

The value of airspeed (u_0), for convenience, is get from the control subsystem where it is defined as a set point.

Tab. 5.3. Data for the Longitudinal

Name	**Symbol**	**Name**	**Symbol**
Cz_u1	C_{zu}	*Cxdeltae1*	$C_{x\delta e}$
Cz_aplha1	$C_{z\alpha}$	*Czdeltae1*	$C_{z\delta e}$
Cz_q1	C_{zq}	*Cmdeltae1*	$C_{m\delta e}$
Cx_u1	C_{xu}	*Wing_Area_S*	S
Cx_alpha1	$C_{x\alpha}$	*Mean_Chord_mc1*	$m\overline{c}$
Cm_u1	C_{mu}	*Iy1*	I_y
Cm_dotalpha1	$C_{m\dot{\alpha}}$	*Mass_m1*	m
Cm_alpha1	$C_{m\alpha}$	*Air_Density_rho1*	ρ
Cm_q1	C_{mq}	*g1*	g

5.4.2. Auxiliary Subsystem

This subsystem is introduced in order to simplify the calculation on other subsystems. Its parameters are listed below:

- $1/I_y$: it corresponds to the inverse of momentum of inertia related to the y-axis;
- $1/m$: it corresponds to the inverse of the airplane mass;

- C_{wzero} : it is an adimensional coefficient calculated through the ratio between the inertial force (weight) and the dynamic pressure multiplied by the wing surface ($\frac{1}{2}\rho v^2 S$);

The Fig. 5.10 exemplifies the subsystem.

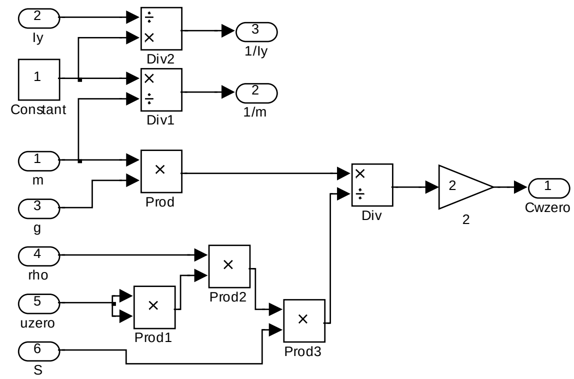

Fig. 5.10. Schema of Auxiliary Subsystem

5.4.3. $X_{\delta p}$ Subsystem

Previously, in section 0, it is known that $X_{\delta p} = 0.3mg$, where *mg* corresponds to the airplane weight.

This subsystem, demonstrated in Fig. 5.11, is responsible to obtain the dimensional control derivative $X_{\delta p}$ through the relation described above.

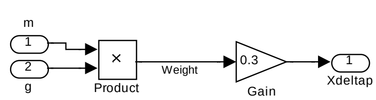

Fig. 5.11. Schema of $X_{\delta p}$ subsystem

Moreover, the Tab. 5.4 contains the inputs and outputs of $X_{\delta p}$ subsystem.

Tab. 5.4. Inputs and Output of $X_{\delta p}$ subsystem

Inputs	*Output*
m – (kg)	$X_{\delta p}$ - dimensional throttle control derivative
g – (ms^{-2})	

5.4.4. $\mathbf{X_{\delta e}, Z_{\delta e}, M_{\delta e}}$ Subsystem

According to section 4.10, the dimensional elevator control derivatives ($X_{\delta e}, Z_{\delta e}, M_{\delta e}$) are calculated following the equations expressed in (4.10,4), which defines the role of this subsystem. These equations are displayed through a block-diagram model, as shows Fig. 5.12.

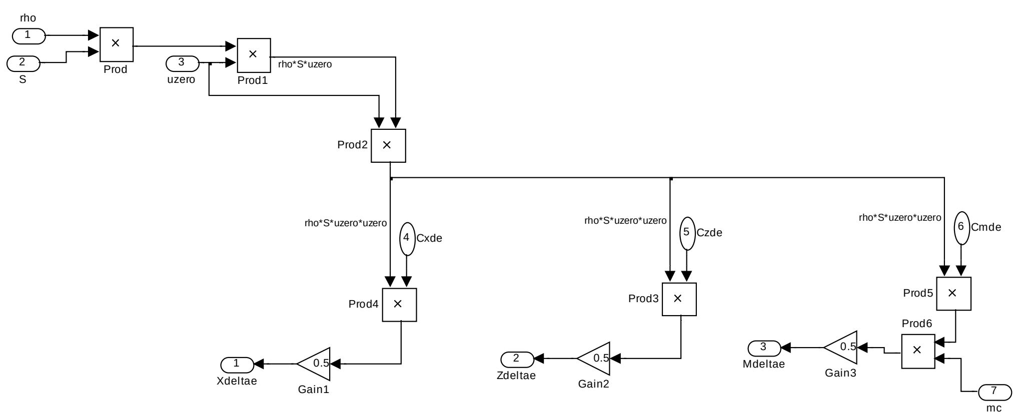

Fig. 5.12. Schema of $X_{\delta e}, Z_{\delta e}, M_{\delta e}$ subsystem

The necessaries inputs for the calculation are listed below:

- *rho (ρ), S, uzero (u_0), mc ($\bar{c}$)*: they correspond to the air density, the wing area, the airplane speed, and the average wing chord;

- $C_{x\delta\delta}, C_{Z\delta\delta}, C_{m\delta\delta}$: they correspond to the nondimensional stability derivatives;

5.4.5. $\mathbf{X_u}$, $\mathbf{X_w}$, $\mathbf{Z_u}$, $\mathbf{Z_w}$ Subsystem

The objective of this subsystem is to calculate the dimensional stability derivatives - X_u, X_w, Z_u, Z_w. Successfully, they are obtained using the relationships listed on Tab. 4.4, and they are developed through the block-diagram method, as is shown on the Fig. 5.13 below.

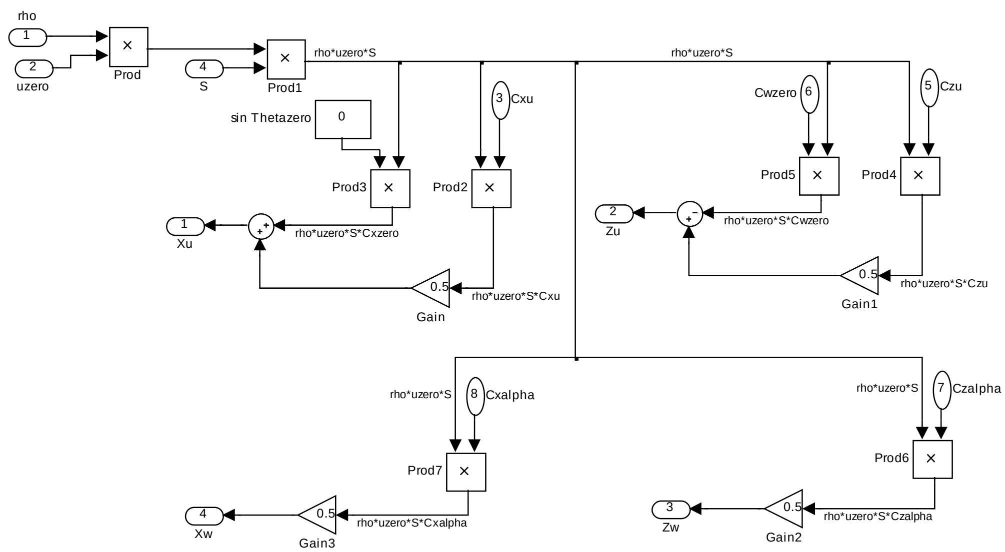

Fig. 5.13. Schema of X_u, X_w, Z_u, Z_w subsystem

It is important to remind that $\theta_0 = 0$ (the airplane started flying parallel to its final trajectory).

The inputs necessaries for the calculation are listed below:

- *rho (ρ)*, *S, uzero (u_0)*: they correspond to the air density, the wing area and the airplane velocity;
- $C_{xu}, C_{zu}, C_{x\alpha}, C_{z\alpha}, C_{w_0}$: they correspond to the nondimensional stability derivatives;

5.4.6. $\mathbf{M_u, M_{\dot{w}}, M_q, Z_q}$ Subsystem

Similar to these dimensional stability derivatives listed above, $M_u, M_{\dot{w}}, M_q$ and Z_q are also calculated using the equations disposed on Tab. 4.4. The Fig. 5.14 shows the block-diagram model developed for this subsystem.

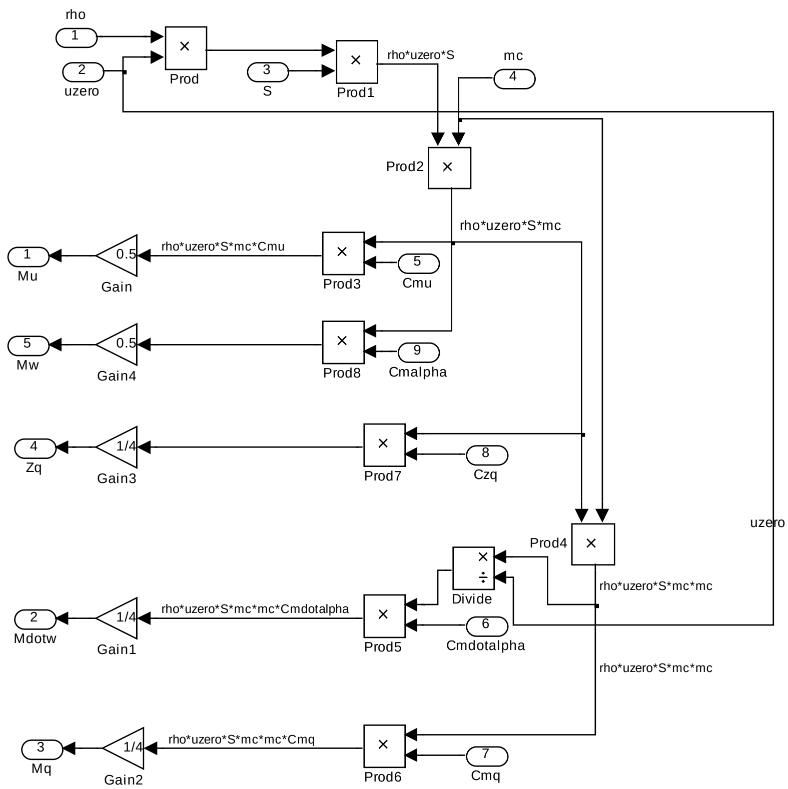

Fig. 5.14. Schema of $M_u, M_{\dot{w}}, M_q, Z_q$ subsystem

In the following list are displayed all necessaries inputs utilized in this subsystem.

- *rho (ρ), S, uzero (u_0), mc ($\bar{c}$)*: they correspond to the air density, the wing area, the aircraft speed, and average wing chord;

- $C_{mu}, C_{mq}, C_{m\alpha}, C_{m\dot{\alpha}}, C_{zq}$: they correspond to the nondimensional stability derivatives;

5.4.7. Z_e Equation Subsystem

This subsystem is aimed to calculate the distance of the aircraft from to the planned flight altitude on z-axis. As it is a downward vertical axis, we find the altitude through $\Delta h = -\Delta z_e$.

Knowing that $\theta_0 = 0$ and using the equation which describes the z-motion given in Eq. 4.45, it results in the following relation:

$$\Delta \dot{z}_E = w - u_0 \Delta\theta$$

The Fig. 5.15 demonstrates it, by using block-diagram method.

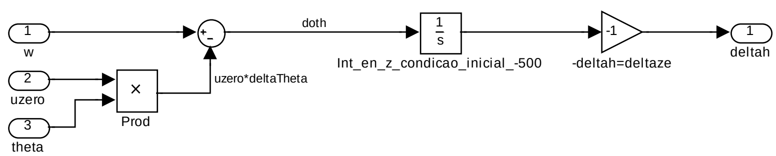

Fig. 5.15. Schema of Z_e Equation Subsystem

It is important to remark that, the initial altitude, placed on z-axis, is defined inside the integrator element.

The inputs for this subsystem are listed below:

- *w, uzero (u_0), theta (θ)*: they correspond, respectively, to the vertical aircraft velocity in F_B, aircraft speed and its pitch angle;

5.4.8. W Equation Subsystem

Responsible to calculate the vertical speed of airplane expressed on F_B and related to the earth frame (placed on the reference trajectory), this subsystem is represented through block-diagram model and displayed on the Fig. 5.16.

The governed equation for this calculation appears in Eq. 4.45 and it is represented below, as

$$\dot{w} = \left(\frac{Z_u}{m}\right)\Delta u + \left(\frac{Z_w}{m}\right)w + \left(\frac{Z_q + mu_0}{m}\right)q + \left[\frac{Z_{\delta_e}}{m}\right]\delta_e$$

Reminding that, thrust engine does not have any component effort in z – direction, thus $Z_{\delta p} = 0$; and it is assumed, in chapter 4, that $Z_{\dot{w}} = 0$.

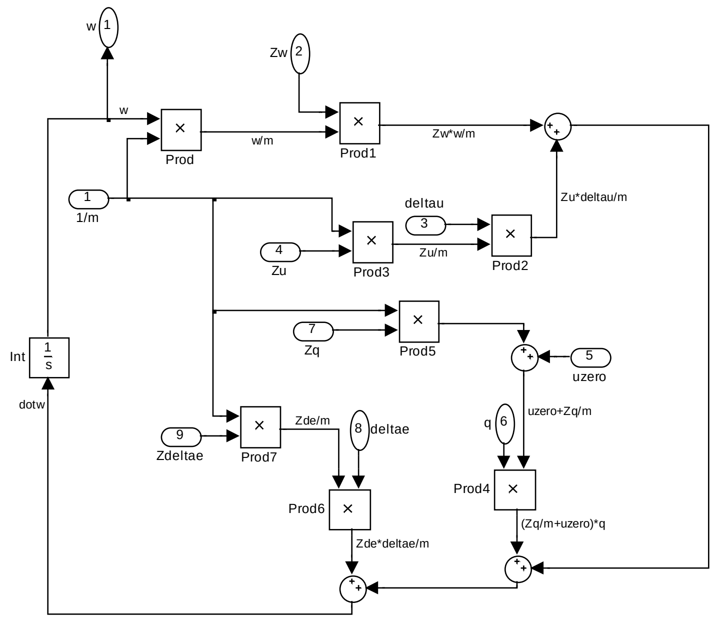

Fig. 5.16. Schema W Equation Subsystem

All necessaries inputs are displayed in Fig. 5.16, and listed as below:

- *1/m*, *uzero (u_0)* : they correspond to physic parameters about the aircraft and the flight;
- Z_u, Z_w, Z_q, $Z_{\delta e}$: they correspond to the dimensional stability derivatives;
- *deltau (Δu)*, δ_e, *q*: they are the flight parameters calculated from other internal airframe subsystems, except for δ_e which is a control parameter done by the control subsystem;

5.4.9. Δu Equation Subsystem

Aimed to find out the aircraft velocity variation (Δu), this subsystem is given with the equation Eq. 4.48 and, jointly, with Eq. 4.55, as it is shown below

$$\Delta\dot{u} = \left(\frac{X_u}{m}\right)\Delta u + \left(\frac{X_w}{m} - \Gamma\right)w + \left(-g + \Gamma u_0\right)\Delta\theta + \left(\frac{X_{\delta e}}{m}\right)\delta e + \left(\frac{X_{\delta p}}{m}\right)\delta p$$

Note that: Γ, named as *DeltaWind* in the block-diagram model (see Fig. 5.17), corresponds to the derivative of the vertical wind profile respect to z-direction and it is defined at the flight reference altitude, as it was explained in section 4.8.

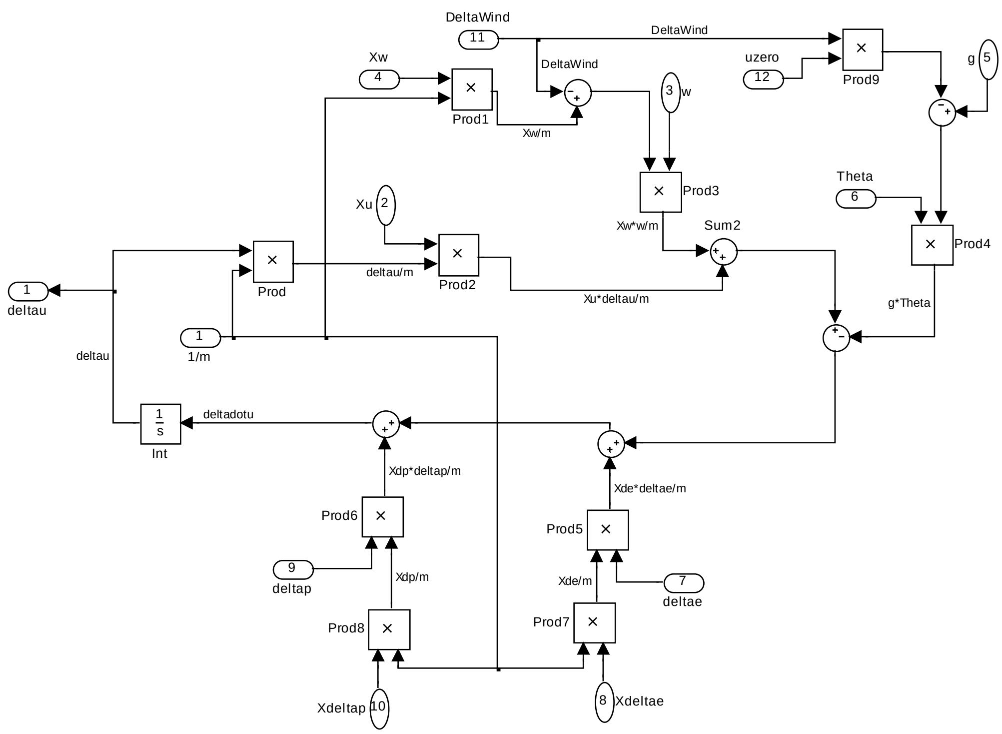

Fig. 5.17. Schema of Δu Equation Subsystem

The inputs for this subsystem are listed below:

- *1/m*, *uzero (u_0), g* : they correspond to the physic parameters about the aircraft and the flight;
- $X_u, X_w, X_{\delta p}, X_{\delta e}$: they correspond to the dimensional stability derivatives;

- *DeltaWind (Γ):* it correspond to dimensional wind derivative;
- *w, Theta (θ),* δ_e, δ_p: they are the flight parameters calculated from other internals airframe subsystems, except for δ_e and δ_p which are control parameters done by the control subsystem;

5.4.10. q Equation Subsystem

Similar to others motion equations, q *Equation Subsystem* is destined to find out the value of pitch speed (q) which is related to frame F_E and represented in F_B. It is administrated by the following equation extracted from Eq. 4.45 and represented in Fig. 5.18, by using the block-diagram method.

$$\dot{q} = \frac{1}{I_y}\left[M_u + \frac{M_{\dot{w}} Z_u}{m}\right]\Delta u + \frac{1}{I_y}\left[M_w + \frac{M_{\dot{w}} Z_w}{m}\right]w + \frac{1}{I_y}\left[M_q + \frac{M_{\dot{w}}\left(Z_q + m u_0\right)}{m}\right]q \ldots$$
$$+\left[\frac{M_{\delta e}}{I_y} + \frac{M_{\dot{w}}}{I_y}\frac{Z_{\delta e}}{m}\right]\delta_e$$

To simplifying the operation, I_y multiplies all terms in the above equation.

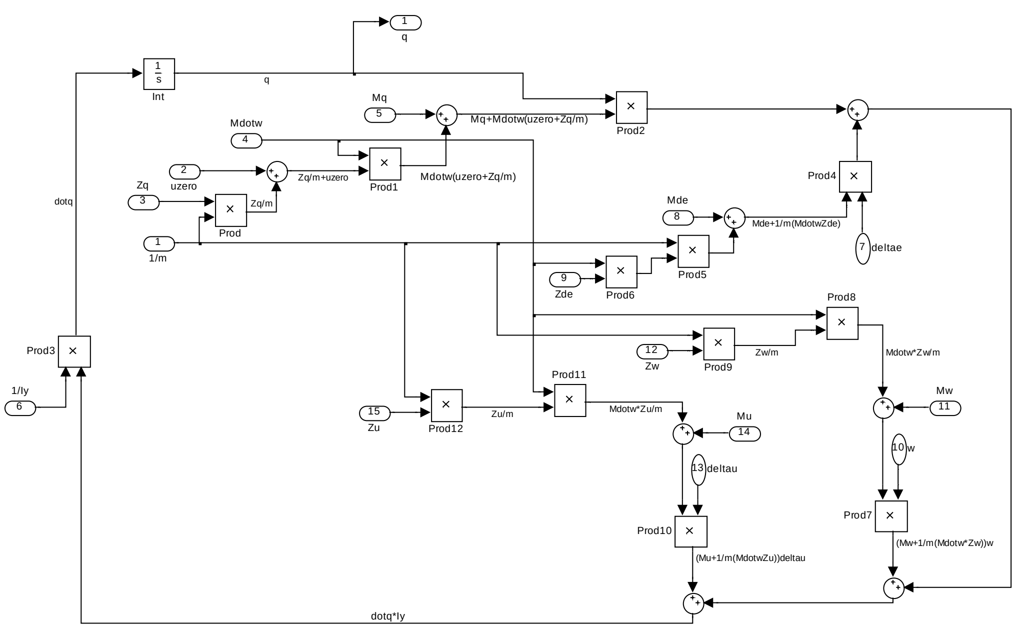

Fig. 5.18. Schema of q Equation Subsystem

The inputs applied, in order to get out the fight parameter *q*, are:

- *1/m*, *1/I*$_y$*, uzero* (u_0): they correspond to the physic parameters about the aircraft and the flight;
- $Z_u, Z_w, Z_q, Z_{\delta e}, M_u, M_w, M_{\dot{w}}, M_q, M_{\delta e}$: they correspond to the dimensional stability derivatives;
- *deltau (Δu), w,* δ_e: they are the flight parameters calculated from other internal airframe subsystems, except for δ_e which is a control parameter done by the control subsystem;

5.4.11. $\Delta\theta$ Equation Subsystem

This subsystem is intended to get out the pitch angle θ through the equation in Eq. 4.45, as is shown in the following

$$\Delta\dot{\theta} = q$$

Reminding that, $\Delta\theta$ is found, by integrating the pitch angular velocity in the time domain. Consequently, as it was assumed before in others equations, the initial condition θ_0 entered on the integrator element is equal to zero. The Fig. 5.19 shows its block-diagram model.

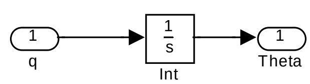

Fig. 5.19. Schema of θ Equation Subsystem

In the next is listed the necessary input for this subsystem. It is,

- *q* : it is a flight parameter calculated from another internal airframe subsystem;

5.4.12. WindData Subsystem

As it own name explains, the *WindData Susbystem* is destined to find out the values of Γ – derivative of the vertical wind profile respect to *z*-direction, and *W* – component of wind velocity in *x*-direction. In others words, is here that is included the necessaries data about the atmosphere condition, in which will be, indirectly, added on the motion equations.

For the first case, similar as it was discussed in section 4.8 and represented in Fig. 4.4, it is assumed a flight under an exponential wind speed profile represented by Eq. 5.3. Thus,

$$W = kh^{0.4} \quad \rightarrow \quad \Gamma = -\frac{dW}{dh} = -0.4kh^{0.4-1} \qquad \text{(Eq. 5.3)}$$

where,

W is the wind velocity, *h* is height above the ground, and *0.4* is the roughness value.

Considering that over the altitude h_∞ , the wind velocity is constant and equal to W_∞; the reference altitude flight is denominated as h_{ref} and it defines W_0 – the reference wind speed.

Knowing these values, *k* and Γ_{ref} are found through the following equation Eq. 5.4.

$$\begin{aligned} k &= \frac{W_\infty}{h_\infty^{0.4}} \\ \Gamma_{ref} &= -0.4\frac{W_\infty}{h_\infty^{0.4}} h_{ref}^{0.4-1} \end{aligned} \qquad \text{(Eq. 5.4)}$$

Noting that, the subscript *ref* refers to the reference flight altitude.

Through the equation Eq. 4.46, Eq. 5.5 calculates the wind speed *W* at any instantaneous height *h*. Thus,

$$W = W_0 + \Gamma_{ref} z_E \quad \rightarrow \quad W = kh_{ref}^{0.4} + \Gamma_{ref}(-h) \qquad \text{(Eq. 5.5)}$$

Moreover, a second case, as is illustrated on the Fig. 5.20, is considered by assuming that the vertical wind velocity profile is a sinusoidal function dependent of the altitude *h*.

$$W = W_\infty \sin\left(\frac{2\pi}{\lambda}h\right) \quad \rightarrow \quad \Gamma = -\frac{dW}{dh} = -\frac{2\pi}{\lambda}W_\infty \cos\left(\frac{2\pi}{\lambda}h\right) \qquad \text{(Eq. 5.6)}$$

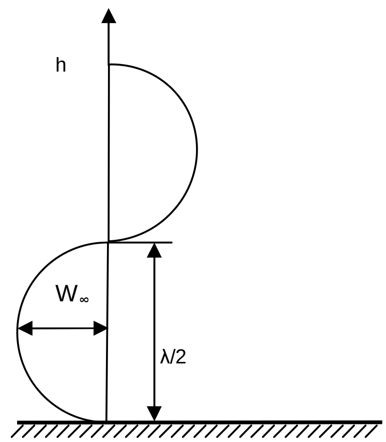

Fig. 5.20. Sinusoidal Wind Profile

The constant value of W_∞ and λ are given later, when the simulation cases will be listed. Similar to the first case, W_0 is the wind speed at the ordered flight altitude given by the reference flight trajectory.

Thus, Eq. 5.7 shows the calculation to find out W for any altitude h.

$$W = W_0 + \Gamma_{ref} z_E \quad \rightarrow \quad W = W_\infty \sin\left(\frac{2\pi}{\lambda}h_{ref}\right) - \frac{2\pi}{\lambda}W_\infty(-h)\cos\left(\frac{2\pi}{\lambda}h_{ref}\right) \qquad \text{(Eq. 5.7)}$$

The Fig. 5.21 and Fig. 5.22 illustrate the block-diagram model for this subsystem with potential and sinusoidal wind velocity profile, respectively.

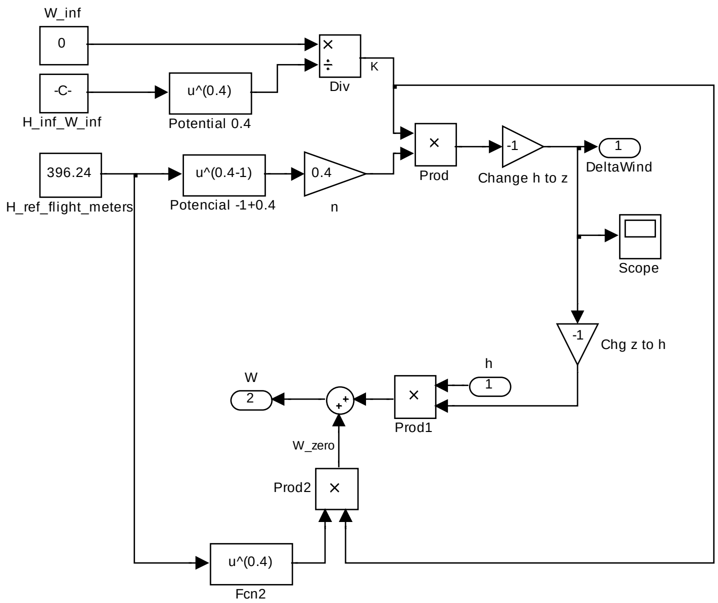

Fig. 5.21. Schema of WinData Subsystem for Exponential Wind Speed Profile

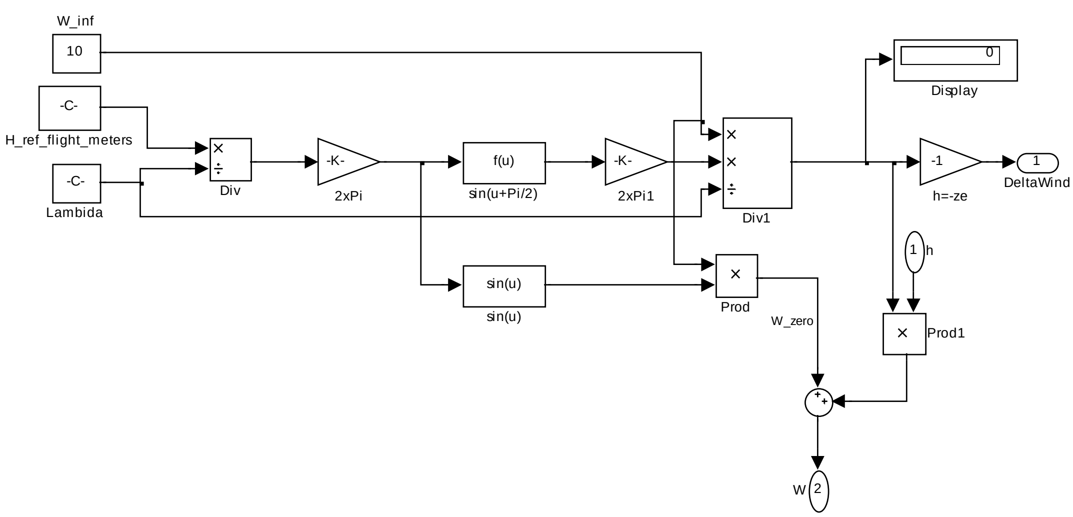

Fig. 5.22. Schema of WinData Subsystem for Sinusoidal Wind Speed Profile

5.5. Cases of Studied Flight Conditions

Once that the model part is done, as it was shown on the development of this chapter, the operating parameters, found out by simulation, define the behavior of the aircraft leaving from a initial flight condition until the reference one.

In the following, it will be defined the initial flight, the reference flight, and also, the atmosphere conditions.

However, it is important to remind those variables which are necessaries to characterize them. They are listed below:

- *Initial Flight Condition:*
    - Initial Altitude – h_{init};
- *Reference Flight Condition:*
    - Reference Airspeed – u_0;
    - Reference Altitude – h_{ref};
- *Atmosphere Condition:*
    - Type I: Exponential Wind Velocity Profile (see section 5.4.12)
        - Maximum Wind Velocity – W_∞;
        - Altitude - h_∞ - in which over it, the wind velocity is maximum and constant;
    - Type II: Sinusoidal Wind Velocity Profile (see section 5.4.12)
        - Maximum Wind Velocity - W_∞;
        - Lambda – λ - in which defines the maximum height where wind velocity profile does not repeat;

Trying to cover the maximum of real possibilities that this model can be utilized, it is considered two “phases” of one flight. These are:

- *Cruising Flight Phase*: airplane flying in high airspeed, and with high altitude – consequently, no interference of wind variation existent near to the soil ($W = cte$) ;

- *Approximation Flight Phase*: always present before the landing operation, it is characterized with small airspeed, when compared to the last case, and small flight altitude which is interfered from the wind variation.

Also, inside each situation is assumed different initial conditions. All of them are illustrate in Table 5.5.

Tab. 5.5. Studies Flight Cases

Case **1 -**		***Cruising Flight Phase***		
	1.1 -	*Over the reference trajectory*		
		Characteristics:	h_{init} =	300 m over h_{ref}
			h_{ref} =	5000 m
			u_0 =	235.9 m/s
			ρ =	0.38 kg/m^3
	1.2 -	*Under the reference trajectory*		
		Characteristics:	h_{init} =	300 m under h_{ref}
			h_{ref} =	5000 m
			u_0 =	235.9 m/s
			ρ =	0.38 kg/m^3

Case **2 -**	***Approximation Phase***		
2.1 -	*Exponential Wind Speed Profile*		
	Characteristics:	h_{init} =	50 m over h_{ref}
		h_{ref} =	396.24 m
		u_0 =	150 m/s
		W_∞ =	10 m/s
		h_∞ =	518.16 m
		ρ =	1.2 kg/m^3
2.2 -	*Sinusoidal Wind Speed Profile*		
	Characteristics:	h_{init} =	50 m over h_{ref}
		h_{ref} =	323.85 m
		u_0 =	150 m/s
		W_∞ =	10 m/s
		λ =	518.16 m
		ρ =	1.2 kg/m^3

In the following chapter, it contains the results obtained from the simulation and its discussions.

6. Results and Analyses

6.1. Previous Information

Many reasons emphasize the use of a longitudinal control in airplanes during on its cruising flight phase and also mainly on the approximation flight one. Importantly, beyond the client satisfaction during the flight, some of these reasons are done for the aircrafts keeping in following the flight trajectory ordered from the navigation system.

This chapter is intended to plot operating parameters in time domain, in which describe the aircraft motion and/or changes on the command parameters carried out by the corrective actions given from control system.

Exemplifying it, for each flight condition studied, as they are presented in section 5.5, the parameters illustrated are: altitude, aircraft speed related to air and to the earth frame, pitch angle, elevator angle and throttle level.

6.2. Results and Analyses

Considering the flights cases assumed in section 5.5, the operating parameters of Boeing 747-100 are:

- ***Cruising Flight Phase:***

*Case **1.1** - Over the reference trajectory*

Time of simulation: 200s

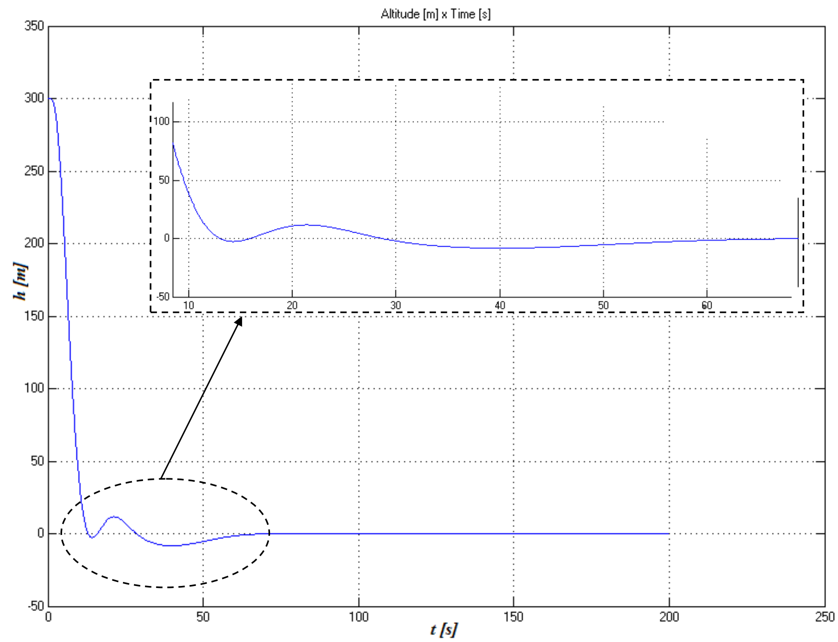

Fig. 6.1. Altitude Variation of Boeing 747-100 under flight conditions of case

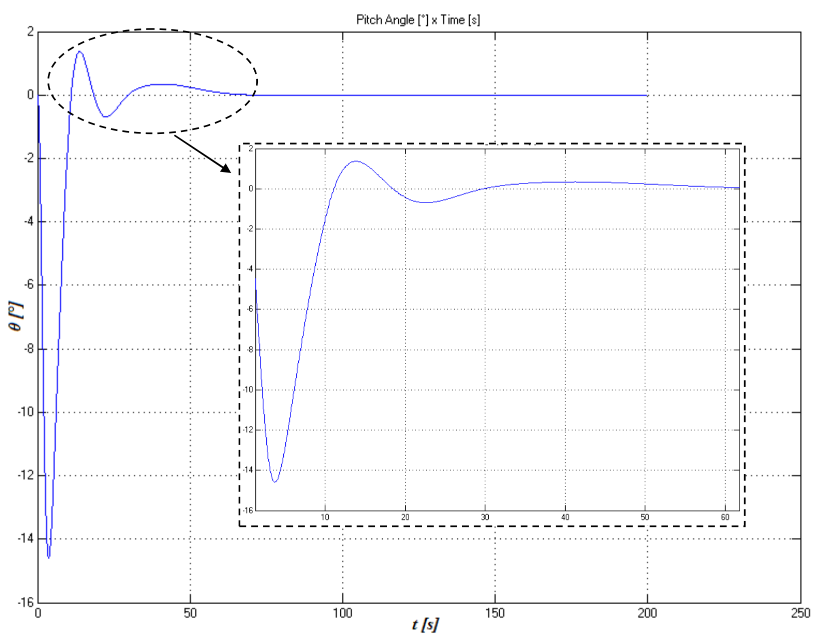

Fig. 6.2. Pitch Angle Variation of Boeing 747-100 under flight conditions of case 1.1

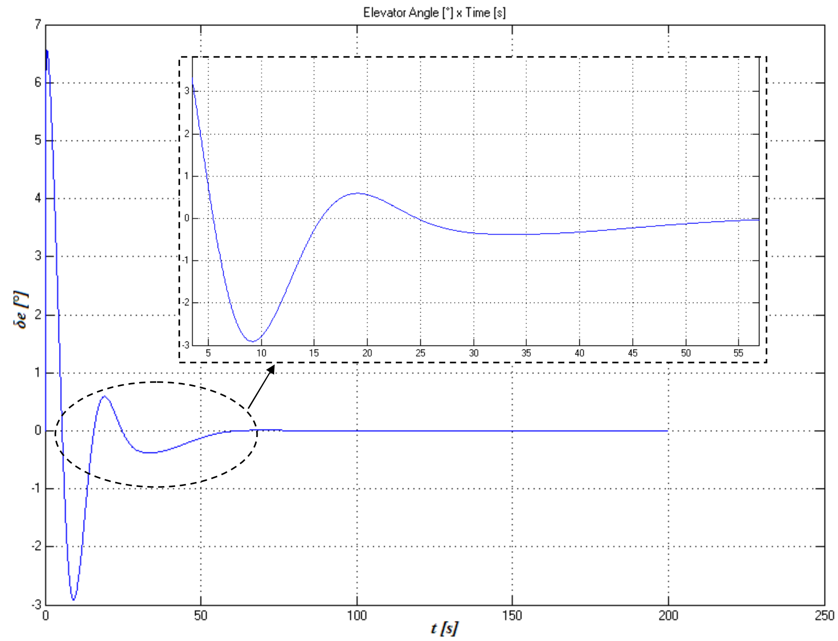

Fig. 6.3. Elevator Angle Variation of Boeing 747-100 under flight conditions of case 1.1

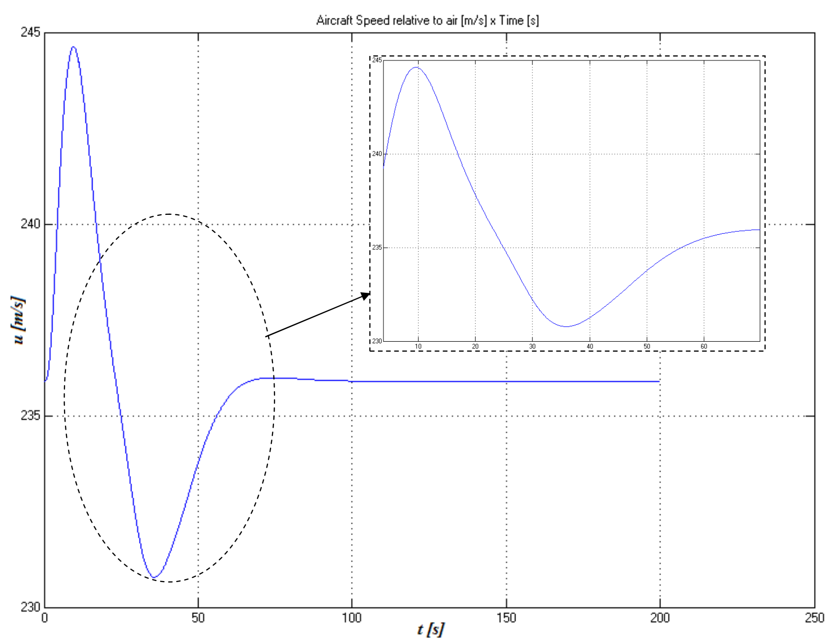

Fig. 6.4. Aircraft Speed Relative to Air of Boeing 747-100 under flight conditions of case 1.1

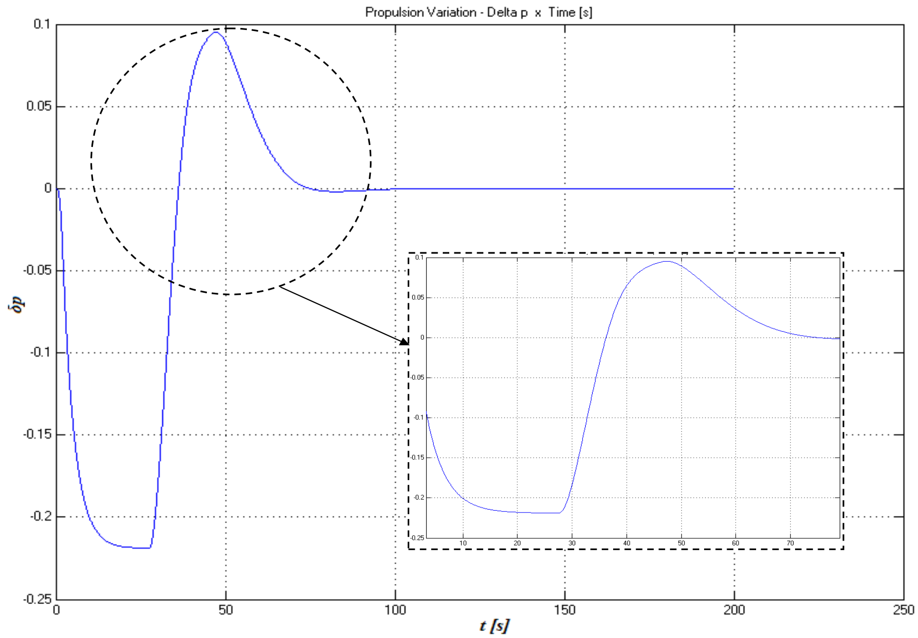

Fig. 6.5. Throttle Level Variation of Boeing under flight conditions of case 1.1

As shown in Fig. 6.1 at the starting time, it coincides with the flight conditions given in case 1.1, which illustrates that the airplane is 300 m over the reference line (*h* = 0). Consequently, the elevator control identifies a new reference pitch angle θ_{ref}, through the constant *K* (see Fig. 5.4), acts on the elevator angle, and finally, on the pitch angle, as illustrate Fig. 6.3 and Fig. 6.2, respectively.

Due to effects of τ_e and momentum of inertia, the pitch angle does not change instantly. Moreover, they influence, also, on the decreasing altitude rate, in which becomes smaller when compared to the ideal case (momentum of inertia and τ_e are negligible).

Through Fig. 6.5, the throttle level is decreasing in the early instants ($0 \le t \le 15$) due to increasing of the airplane speed. This event is provoked by the effects of gravity during the downward motion. It falls until the minimum throttle level of -0.219, which is carried out by the throttle limiter (see Fig. 5.6).

Even closing to the reference altitude at $t \approx 15s$, the system does not "stabilize" (steady-state) on its longitudinal motion because of its control parameters magnitudes.

As shown the Fig. 6.3 for $5 \le t \le 15$, δ_e present a negative value due to the control system, in which verify that the reference pitch angle is smaller than the actual one. However, its corrective action goes through the expected, and the airplane goes over the reference trajectory, as illustrates the Fig. 6.1.

Also, Fig. 6.4 illustrates a decreasing in its airspeed. Therefore, once that this is intended to get the steady state at reference aircraft speed of 235.9m/s, is verified that throttle level is going up.

Following the reference of control system [10], the spending time to get a steady-state are:

- *50s* – for time bigger than it, the altitude is considered in steady-state at reference flight altitude;
- *41s* – for time bigger than it, the airplane speed is considered in steady-state at reference flight speed;

Resuming, for the initial flight conditions described in case 1.1, the airplane is considered in steady-state case, in other words, following the trajectory ordered by the navigation system with the flight reference airspeed, after 50s.

*Case **1.2** - Under the reference trajectory*

Time of simulation: 200s

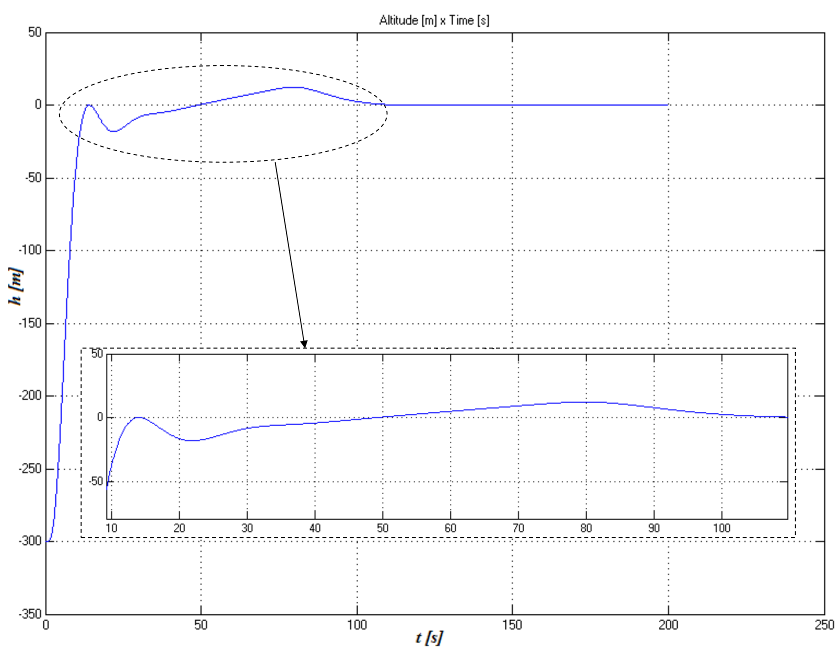

Fig. 6.6. Altitude Variation of Boeing 747-100 under flight conditions of case 1.2

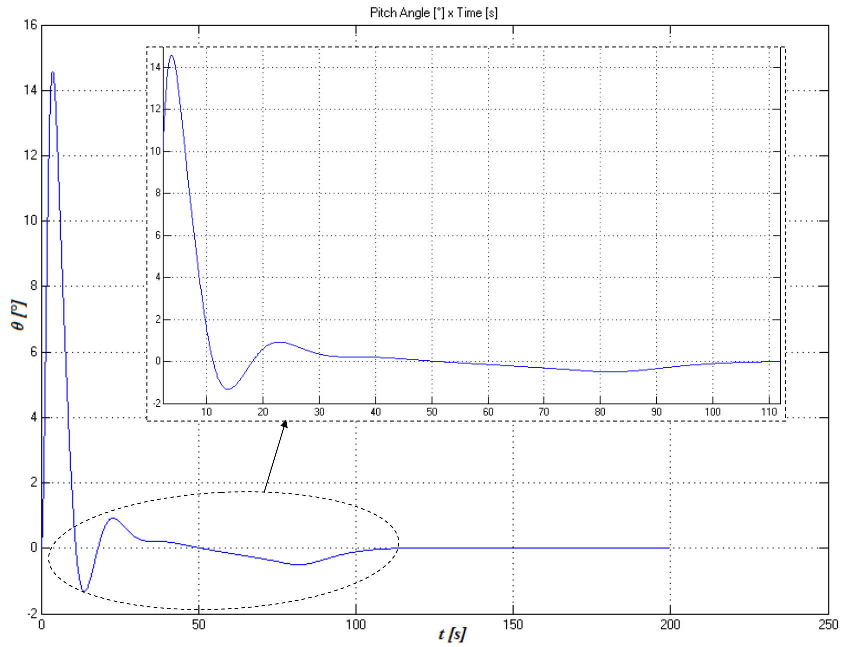

Fig. 6.7. Pitch Angle Variation of Boeing 747-100 under flight conditions of case 1.2

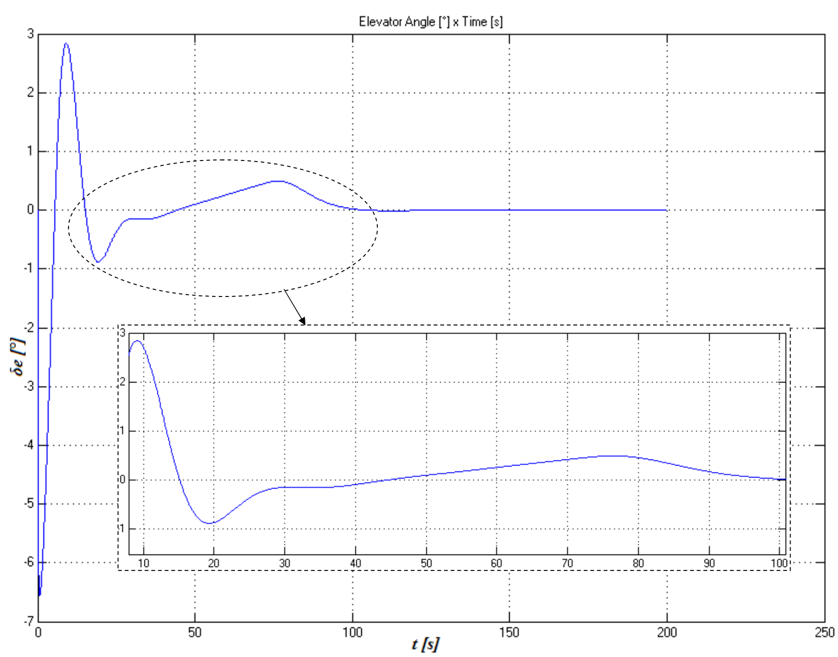

Fig. 6.8. Elevator Angle Variation of Boeing 747-100 under flight conditions of case 1.2

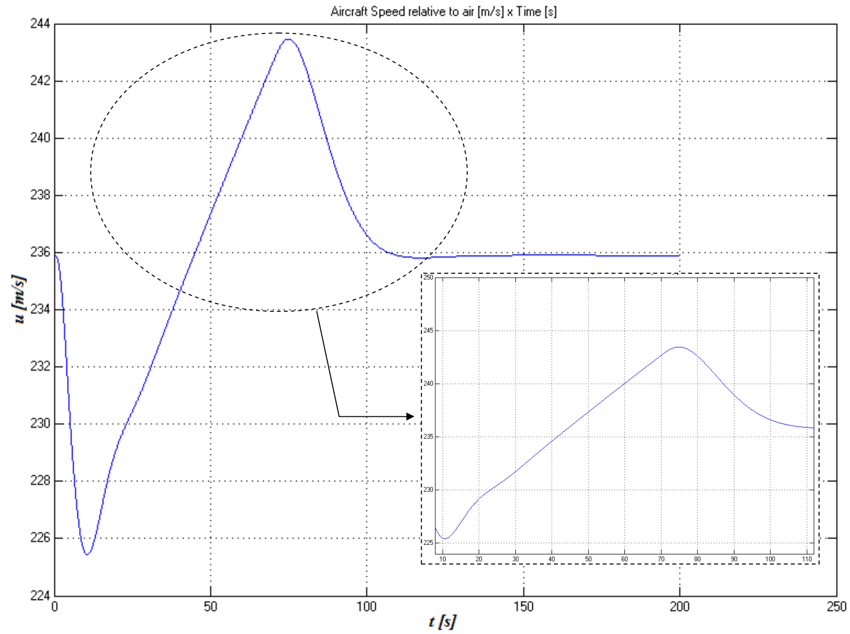

Fig. 6.9 Aircraft Speed Relative to Air of Boeing 747-100 under flight conditions of case 1.2

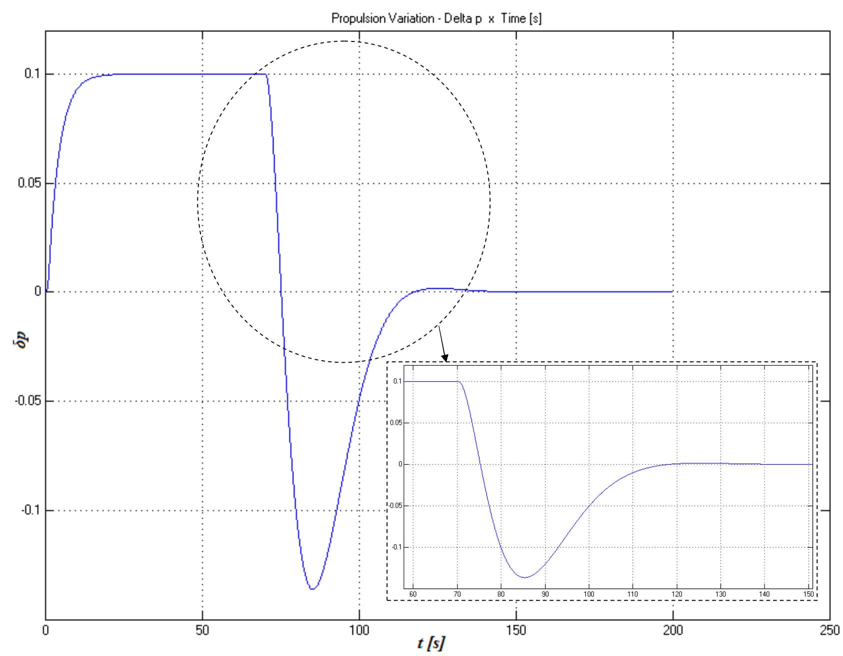

Fig. 6.10. Throttle Level Variation of Boeing under flight conditions of case 1.2

Different from what may be expected between the cases 1.1 and 1.2, the time historic of the operating parameters is not mirror image referenced on *x*-axes placed and at h = 0. For example, the Fig. 6.6 demonstrates it, in case of altitude variation.

As Fig. 6.6 and Fig. 6.9 illustrate, the reduction on the airplane speed, due from the ascend motion, diminishes aerodynamic pressure around all control surfaces of the airplane, and also, consequently, its aerodynamic forces and moments. Moreover, as is shown in Fig. 6.10, the corrective thrust force is restricted in one maximum ordered from the throttle limiter, at *0.1*.

As a result of these effects, the answers from the airplane are slowly.

Similarly as happen in the last case, the control system creates some small oscillations around the steady-state condition, as show the Fig. 6.6.

Following the control system theory [10], the times to get the steady-state flight condition are:

- *93s* - for time bigger than it, the altitude is considered in steady-state at reference flight altitude;
- *86s* - for time bigger than it, the airplane speed is considered in steady-state at reference flight speed;

- ***Approximation Flight Phase:***

<u>*Case **2.1***</u> ***–*** *Exponential Wind Speed Profile*

Time of simulation: 120s

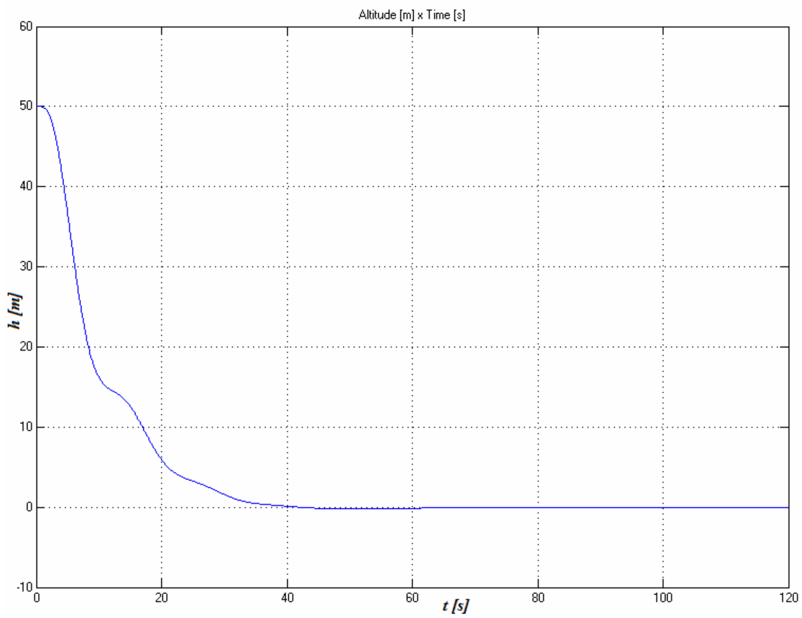

Fig. 6.11. Altitude Variation of Boeing 747-100 under flight conditions of case 2.1

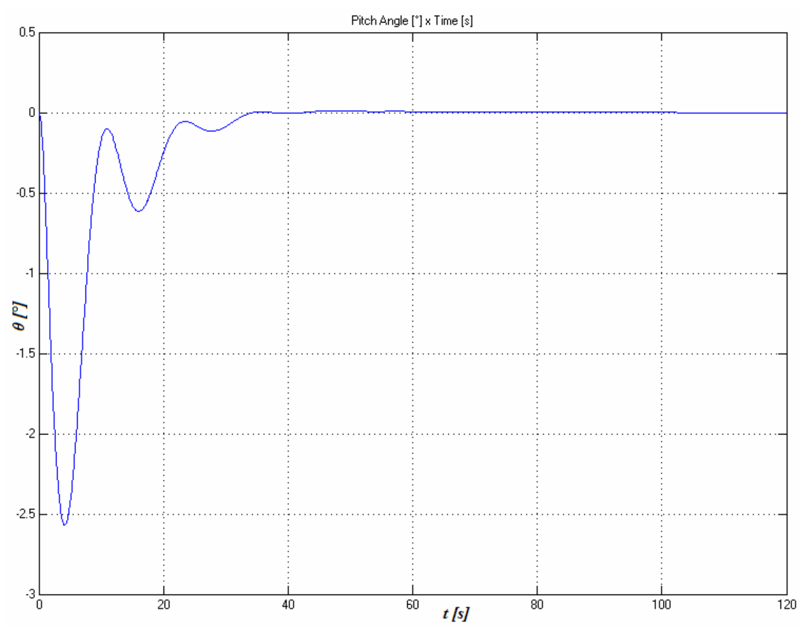

Fig. 6.12. Pitch Angle Variation of Boeing 747-100 under flight conditions of case 2.1

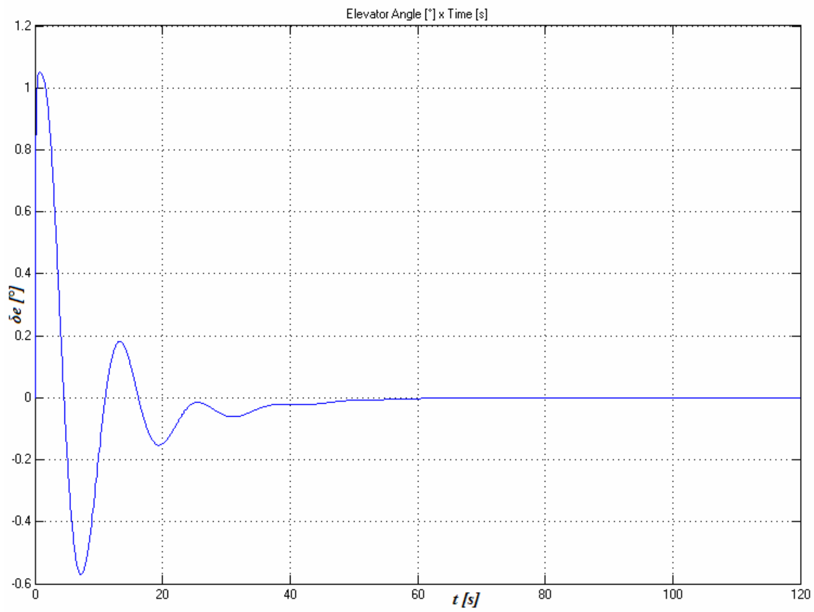

Fig. 6.13. Elevator Angle Variation of Boeing 747-100 under flight conditions of case 2.1

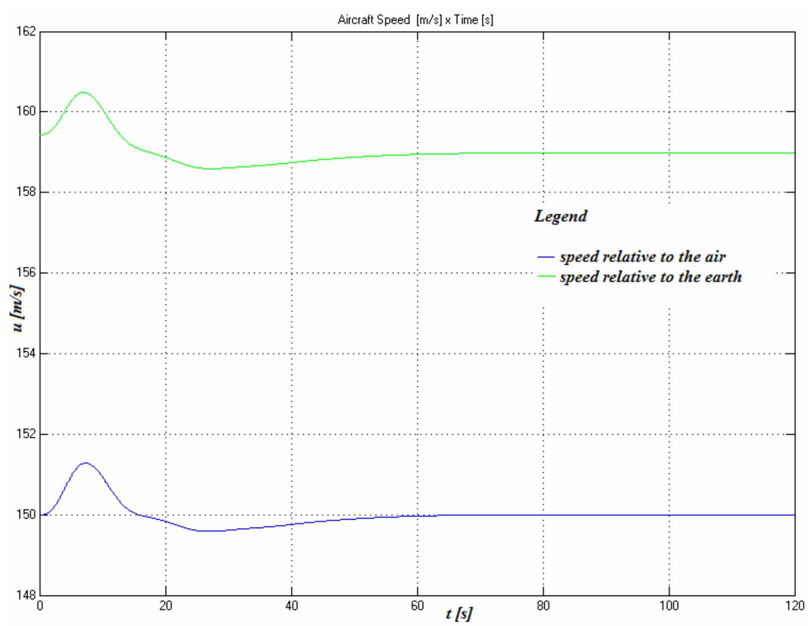

Fig. 6.14. Aircraft Speeds Relative to Air and Earth of Boeing 747-100 under flight conditions of case 2.1

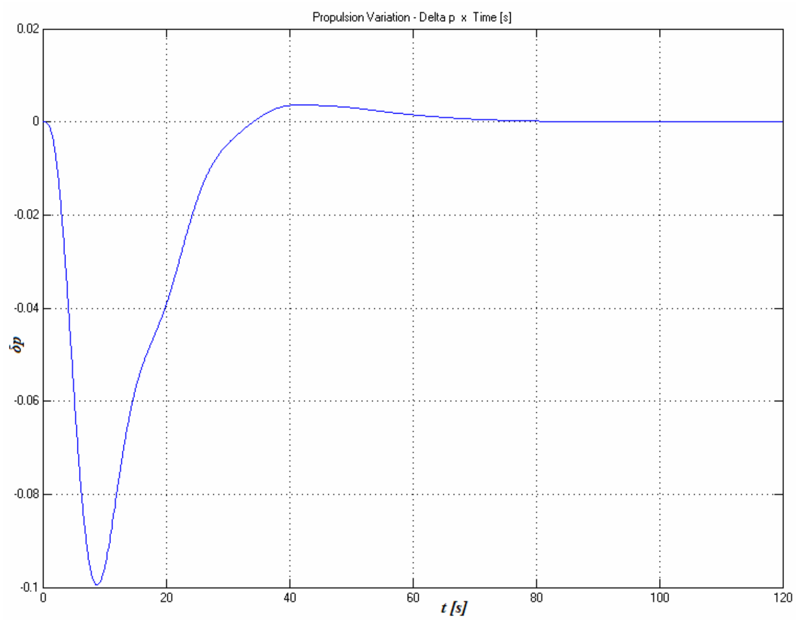

Fig. 6.15. Throttle Level Variation of Boeing under flight conditions of case 2.1

Through the initial flight conditions given in case 2.1, the airplane is placed 150m over the reference flight altitude at *h* = 0, as shows the Fig. 6.11 at *t* = 0.

Purposed to diminish its flight altitude until the reference one, the control system increases the elevator angle in which provides a downward motion through a vertical rotation, as illustrate Fig. 6.11 and Fig. 6.12, respectively.

By that, as its airspeed increases (see Fig. 6.14), is presented a decreasing on the throttle level, through Fig. 6.15.

Remarking that, as wind speed varies exponentially with the flight altitude, the two curves, in Fig. 6.14, are not distanced from a constant value from each other.

At *t* = 10s approximately, the aircraft experience an over increase in θ, in which affects its altitude decreasing rate. The possibilities, which explain this happen, are based on the control system (speed and/or pitch angle control) or from atmosphere condition. The last prospect may influence the system, once is presented an increase in the aircraft speed.

Also, due to high air density occurred in low altitudes, the answers to the control system from the airplane are quickly when compared to cruising flight phase. As a result of it, the airplane motion does not experience considerable oscillations around the reference altitude flight.

Applying the control system theory [10], the times spent for the flight becomes a steady-state one at the reference flight altitude, are:

- *20s* - for time bigger than it, the altitude is considered in steady-state at reference flight altitude;

Remarking that for the airplane velocity, it is considered in steady-state condition during all time inside the range of 150 m/s ± 1%.

*<u>Case **2.2**</u>* ***–*** *Sinusoidal Wind Speed Profile*

Time of simulation: 120s

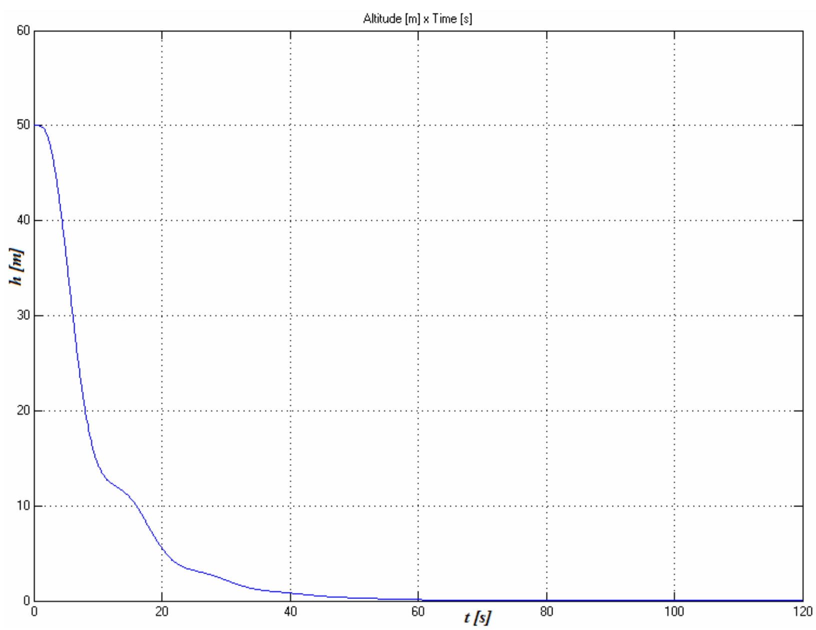

Fig. 6.16. Altitude Variation of Boeing 747-100 under flight conditions of case 2.2

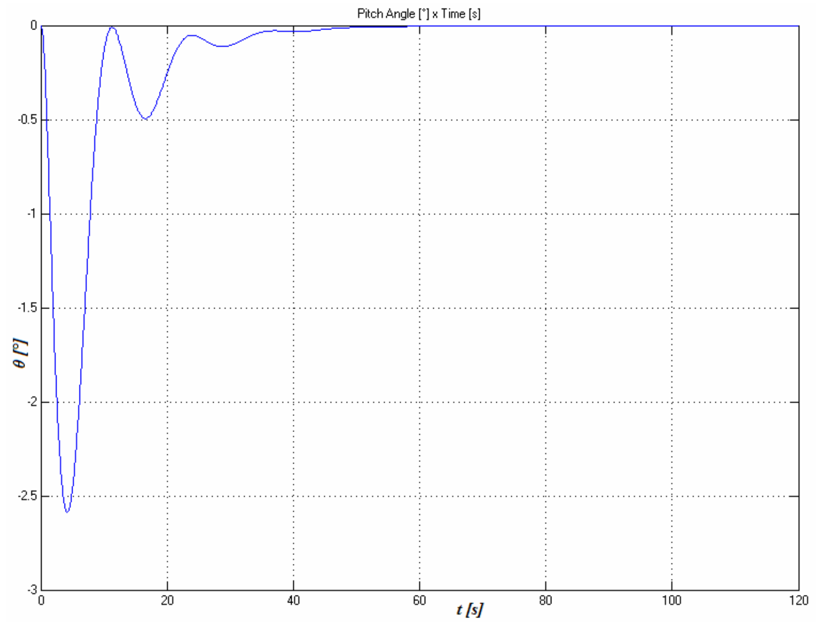

Fig. 6.17. Pitch Angle Variation of Boeing 747-100 under flight conditions of case 2.2

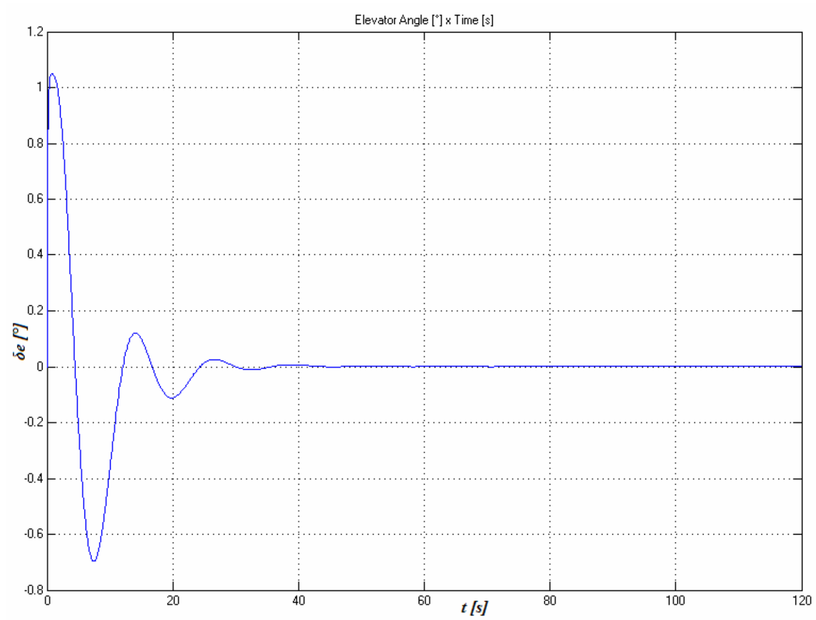

Fig. 6.18. Elevator Angle Variation of Boeing 747-100 under flight conditions of case 2.2

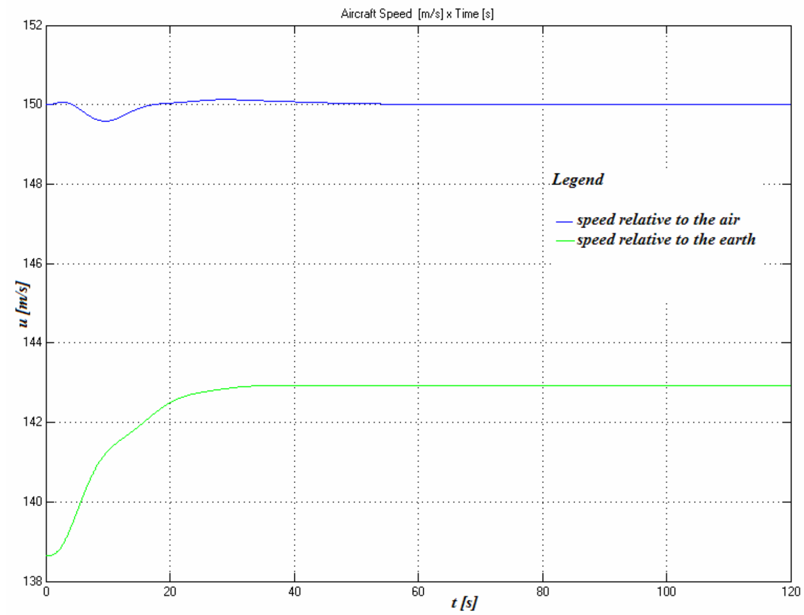

Fig. 6.19 Aircraft Speeds Relative to Air and Earth of Boeing 747-100 under flight conditions of case 2.2

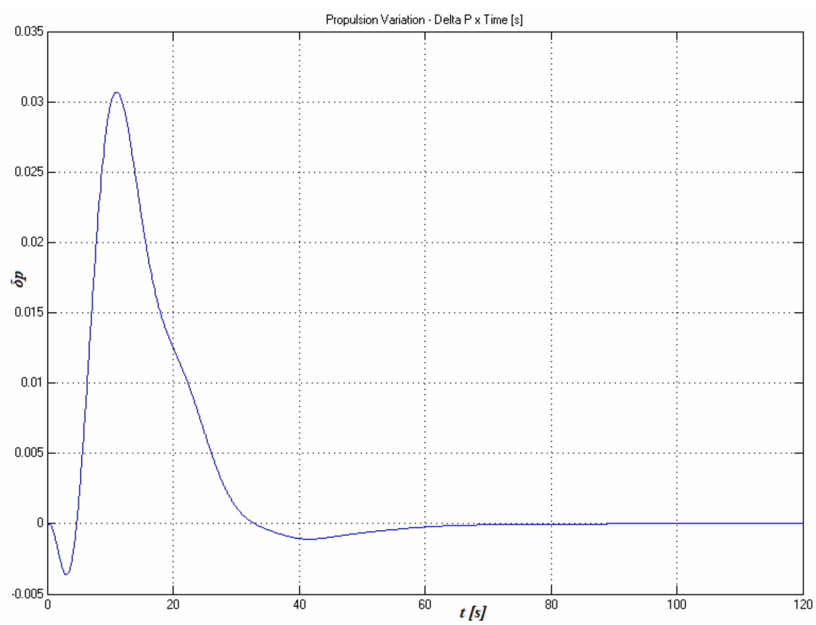

Fig. 6.20. Throttle Level Variation of Boeing under flight conditions of case 2.2

A curious event is happen in this case. Different from all others cases studied with initial altitude over to the reference one, in case 2.2, close to the starting time, the throttle level increases considerable.

What explain this phenomenon are the conditions dictated from atmosphere environment which influence the action of the speed control system. From Fig. 6.19, when compared both speed related to earth and to air, it is marked that the aircraft attacks an opposite wind through time, denominated as head wind. For example, at its initial altitude *h*=50m (see Fig. 6.16), the wind magnitude is 11m/s.

However, on the other hand, pitch angle and elevator angle presents almost the same behavior, characteristic from those which is intended to decreases its flight altitude. The Fig. 6.17 and Fig. 6.18 show this feature.

It is notable, also from these figures, some small oscillations close to the reference flight altitude, usually present in feed-back control system.

Applying the control system theory [10], the times spent for the flight becomes a steady-state one at the reference flight altitude, are:

- *20s* - for time bigger than it, the altitude is considered in steady-state at reference flight altitude;

Remarking that for the airplane velocity, it is considered in steady-state condition during all time inside the range of 150 m/s ± 1%.

Conclusions

It is notable, particularly in Europe, the growth of airline companies and, consequently, the numbers of aircrafts and flights. Going on, a mainly remark should be done to those named as “Low-Cost” due to the easily consumer access of its commodity, efficiency, security and its rapidity, among others, through a, relatively, cheap means of transport.

Generally, the complex, presents on the air traffic control, enlarges proportionally or even exponentially with the number of flights, in manner to keep their viability and safety factor.

However, it becomes a curious and an impressible system in its complexity, when the control goes beyond the traffic field, by controlling itself - the airplane motion on adverse atmosphere and flight conditions - through an *automatic flight control system* which provides a big range of qualities, as for example: a *stability augmentation system*.

As an objective of this present report, the operating flight parameters of the Boeing 747-100, commanded from the control subsystem (described in chapter 5) on different flight conditions, are found out by simulation and they are listed and plotted in chapter 6. In analyzing them based on the dynamic of flight involved, all present the expected behavior.

Nevertheless, some slight differences are noted. Due to changes on the physical properties of the air, the *case* 1, when compared to *case* 2, presents small spending time to get the steady-state condition. In other words, the responses, get from the same variation on the control surfaces, are lower in high altitudes.

Complementary, is not verified similarities in answers between *cases 1.1* and *1.2* which, initially, are symmetric when related to the ordered flight trajectory. This event is due to the limiter throttle level. It has a bigger action time on the flight *case* 1.2, where the airplane is placed under the target trajectory.

However, it is remarked that, due to the restriction of the aircraft model by using the linearized motion equations on its longitudinal behavior, only small disturbances can be applied.

For future works, an interest result may be obtained by simulating a complete flight, where the operating flight parameters are plotted throughout all the flight phase, from taking-off until the landing process, and in all motion ways: longitudinal, lateral and rolling motion.

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Leonardo Sanches

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